PARTIALLY DIRECTED SEARCH IN THE LABOR MARKET

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I study the labor market implications of limited information inherent in the job search process. Workers pay a cost to direct job search that is proportional to the divergence between the chosen search strategy and a benchmark random search strategy. With this cost, workers apply to every job with a positive probability, but apply to high-payoff jobs with higher probabilities. In a wage posting model with partially directed search, employers have monopsony power: firms extract a markdown due to the cost of directing search. Efficiency of the market equilibrium depends on whether the markdowns are equally distributed across firms. Inefficiency arises when search cost is intermediate, which has new implications on policy remedies to monopsony.

Keywords: Competitive Search; Labor Market; Monopsony.

1. INTRODUCTION

Information is crucial for job search. Workers need to know the full set of relevant information in order to find their best matches, including the wage that each job offers and the odds of getting hired. However, assuming that workers have full access to this information is unrealistic. Only a small fraction of job postings contain explicit information regarding wages (e.g., Marinescu and Wolthoff, 2016; Banfi and Villena-Roldán, 2019). On the other extreme, assuming that workers do not have any information and randomly search for jobs is also unrealistic. For example,

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high wage vacancies attract more applicants[1] (e.g., Belot et al., 2018). The degree to which workers can direct their search matters for our understanding of the competition among employers for applicants and the allocation of workers across firms with different productivities. Equilibrium search theory assumes that workers have either full information (directed search) or no information (random search) about the relevant characteristics of jobs. Economists lack a tractable equilibrium framework to the implications of partial information on wages and allocations.

This paper provides an equilibrium framework to study the implications of partial information on wages, efficiency, and policies.[2] In the model, firms post wages to attract applications. Workers maximize expected payoffs by choosing the probability of applying to different firms. When deviating from a random search strategy, workers need to pay a cost proportional to the distance between their chosen strategy and random search strategy, measured by the Kullback-Leibler divergence between these two distributions. Workers trade off the benefit of applying to better jobs against the cost of directing search. Equilibrium search strategy is partially directed: workers apply to better jobs with higher probability, but also apply to all jobs with positive probability. The per-unit cost of directing search governs how directed job search is. This one parameter allows the model to accommodate random search, directed search, and in-between cases.

This paper first makes a theoretical contribution: I embed limited information into the competitive search equilibrium, to study the interaction between search friction and information friction. I start with a wage-posting game with costly directed search, between finite workers and firms. This simple model extends the analytical approach of Burdett et al. (2001) to an environment with costly directed search. I prove the existence and uniqueness of the symmetric subgame equilibrium, and its convergence to a limiting equilibrium with competitive features, the *entropic competitive search equilibrium*.

[1]The elasticity between applicant-per-vacancy to wage is 0.7 to 0.9.

[2]Cheremukhin et al., 2020 uses the term *targeted search* for search strategy with entropy cost; I use the term *partially directed search* to highlight the link between the search equilibrium in this paper and the study of partially directed search equilibrium in literature (e.g., Menzio, 2007)

This equilibrium concept generalizes the notion of a *competitive search equilibrium* into richer information settings. In the limiting equilibrium, (i) firms maximize their profit given the equilibrium market utility of workers and worker's optimal search decision; (ii) workers make their optimal search decision given the equilibrium wages and job finding probabilities at different firms; (iii) the applicants to all firms add up to the exogenous measure of workers in the economy. The analytical tractability of the competitive search equilibrium also prevails in the entropic competitive search equilibrium. Specifically, looking for an equilibrium is equivalent to looking for a market price that equalizes the demand and supply of applications. In the unique limiting equilibrium, both the allocations and wages can be solved in closed form.

Utilizing the newly developed framework, I study the implications of limited information on the equilibrium wage setting in the labor market. The firms in an entropic competitive search equilibrium face an upward-sloping supply curve of applications, due to both the information friction and the search friction. The cost of directing search governs the competition among firms, by changing workers' ability to compare alternatives. Costly directed search thus generates monopsony power even when firms are infinitesimal relative to the market: firms extract a markdown that is increasing in the cost of directing search in equilibrium.

Next, I study the allocative consequences of the monopsony power due to limited information. A social planner that maximizes total output net of the cost of directing search will value each applicant by her contribution to matches minus the marginal cost of directing search. The markdown due to the cost of directing search creates a wedge between the marginal value of applying to a firm for workers in equilibrium and their social values. The market allocation is efficient only when the markdowns are evenly distributed across firms. When the cost of directing search is high enough, the unproductive firms are bounded by the outside option of workers and extract less markdown than the productive firms. In this case, workers apply to the unproductive firms too often compared with the efficient allocation. The impact of the cost of directing search on efficiency is non-monotonic, because the the dispersion of markdowns matters less for allocations when the cost increases, although the dispersion itself rises with the cost. In fact, in the two extremes of random and directed search, the market equilibrium is efficient. The cost of directed search only generates inefficiency in the intermediate cases.

The optimal redistribution policy should balance the markdowns across firms. A minimum wage increases the lowest wage that is feasible, and forces the unproductive firms to extract a lower markdown. The increased markdown dispersion among firms induces workers apply to the unproductive firms even more often than the market equilibrium. As a result, a minimum wage hike worsens the distortion due to monopsony. Meanwhile, it increases employment by reallocating job applicants from the firms with a lower job finding probability to the firms with higher job finding probability. An alternative policy instrument, corporate profit taxation, can achieve the goal of redistribution and efficiency simultaneously. A progressive corporate profit tax discourages firms from posting low wages, and more so for the productive firms. In doing so, the progressive corporate taxation decreases the markdown dispersion between firms, and increases the efficiency of the market allocation.

The equilibrium in the baseline environment with costly directed search is equivalent to an environment where workers do not directly observe wages, but can acquire information about wages at a cost. The cost function with Kullback-Leibler divergence can be derived from models of information acquisition through rational inattention (Sims, 2003). In the environment with rational inattention, workers do not observe the wage posted by firms and need to learn about the payoff of applying to different firms. The cost of learning is proportional to the reduction of uncertainty from the prior distribution to the posterior distribution of payoffs. A search equilibrium with rational inattention is simply a collection of equilibria with observed wage profiles and the cost of directed search, indexed by the productivity profile of firms. As a result, although the model is motivated by limited information in job search, I can focus on costly directed search with observed wages.

I discuss three issues around the baseline partially directed search model. (i) I discuss the implications of the partially directed search on the labor market impacts of the improved information technology. (ii) I discuss two approaches to identify this cost given different datasets. (iii) I show the baseline model can be easily extended to a general class of divergence measure called the f-divergence.

This paper is organized as following: the environment and characterization of the finite game are in section 2; the environment and characterization of the limiting equilibrium are in section 3; efficiency of the market equilibrium and its policy implications are in section 4; I omit the micro-foundation of the partially directed search model to appendix B; The discussion on the information technology and quantification is in 5 and the extension to generalized cost function is in appendix C. I provide the link between partially directed search model to the monopsony models with amenities in appendix D; The proof is omitted from main text and included in appendix E.

Literature and Contributions. This paper is related to research in search theory, the study of labor market and the study of bounded rationality. Search theory is based on the premise of the lack of information (Stigler, 1961). The search literature has developed along two lines of research that assume either full information or no information. The intermediate case with partial information is less studied (Wright et al., 2019).

Random search models assume searchers do not have information regarding who to meet. Workers search, meet, and then decide whether to match (e.g., McCall, 1970). In an equilibrium with random search (e.g., Mortensen, 1970, Mortensen and Pissarides, 1994, Shimer and Smith, 2000, Postel-Vinay and Robin, 2002, Elsby and Michaels, 2013, Bagger and Lentz, 2018), wages are determined by an exogenous bargaining block. Information, search frictions, and terms of trade interact through outside options of agents. In these models, market power is a primitive: firm's market power is their bargaining power.

Competitive search models assume perfect information and the ability to choose. In a competitive search equilibrium, workers decide who to meet, search, then decide whether to trade (e.g., Montgomery, 1991, Shimer, 1996, Moen, 1997, Acemoglu and Shimer, 1999, Burdett et al., 2001, Shi, 2001, Shimer, 2005, Eeckhout and Kircher, 2010, Menzio and Shi, 2010, Guerrieri et al., 2010, Kaas and Kircher, 2015, Schaal, 2017). In these environments, the interaction between search friction and terms of trade is the main focus; information does not enter this interaction.

This paper starts from a game-theoretic environment among finite numbers of workers and firms, a methodology from Peters (1997), Burdett et al. (2001), and Galenianos and Kircher (2012). In the limiting case where population grows to infinity, it provides a tractable framework to study how information, search friction, and rent sharing interact. It highlights the implications of partial information on efficiency of a search equilibrium, and hopefully start a new avenue for future studies of frictional markets where information friction plays a vital role.

There are alternative ways to consider flexible information in a search model. Burdett and Judd (1983), Burdett and Mortensen (1998), and Acemoglu and Shimer (2000) consider an environment where a fraction of searchers can compare two offers. In recent work such as (Lester, 2011, Choi et al., 2018, Bethune et al., 2019), a fraction of searchers are informed and direct their search; others are uninformed and randomly search. Michelacci and Suarez (2006) considers an environment where firms can endogenously choose whether to post or negotiate wages. This paper differs by considering endogenous information friction, coming from the rational inattention of workers. This flexible learning process lends to tractability, even with both endogenous search friction and with firm heterogeneity, which is hard to incorporate in the existing models.

This paper is directly linked to the rising interest in partially directed search behavior. Menzio (2007) studies a cheap talk theory of partially directed search. When productivity is private information, productive firms use noisy signal to hide their types and gain a better position in bargaining. Cheremukhin et al. (2020) applies the same entropy cost to a matching model where the payoffs of matches are exogenous. They use the the entropy cost function to study how the resulting matching decisions affect sorting between males and females. Pilossoph (2012) and Lentz and Moen (2017) applies the reduced-form Logit decision rule to models with Nash bargaining. In their study, wages are either exogenously given (Cheremukhin et al., 2020) or negotiated outside of search decisions (Pilossoph, 2012 and Lentz and Moen, 2017). This paper studies a search equilibrium where both wages and allocations are endogenous.

By doing so, this paper nests the random and the competitive search models as special cases, and is suitable for issues such as wage dispersion and monopsony power.

This paper also contributes to the study of firm market power. The idea that lack of information leads to market power traces back to the Diamond Paradox (Diamond, 1971): if all consumers randomly search for deals, then any positive switching cost leads to monopolistic pricing. This paper nests the Diamond (1971) case as a special case when the cost of directing search is high, and also brings attention to the static competition between hiring firms. In addition, the decision rule when search friction vanishes resembles the decision rules from monopsony models with amenities, which is used in papers such as Card et al. (2018), Berger et al. (2019), and Lamadon et al. (2019).

Lastly, this paper contributes to the study of bounded rationality. It extends the study of rational inattention to an equilibrium with search frictions. This paper builds the literature that study the decision implications of rational inattention (e.g., Caplin and Dean, 2015, Matéjka and McKay, 2012, Matějka and McKay, 2015, Ravid, 2019). There is a long literature integrating bounded rationality with macroeconomic models (e.g., Sims, 2003, Woodford, 2009, Abel et al., 2013, Alvarez et al., 2017, Molavi, 2019), most of which have been focused on how information frictions lead to inertia of adjustments and amplification of aggregate shocks. This paper focuses on how sparsity of information dampens competition in the market and generates rents for the market organizers. This paper also incorporates search friction and extending the finite environment to a limiting economy. Characterizing a finite equilibrium with rich heterogeneity and rational inattention is not a trivial task, due to the strategic interaction between firms. This paper provides a limiting equilibrium concept that captures the essential mechanism of bounded rationality, but remains tractable.

2. PARTIALLY DIRECTED SEARCH IN FINITE ECONOMY

2.1. 2×2 *Economy*

Setup. – The economy has two workers and two firms. Workers are indexed by $i = 1, 2$, and firms are indexed by $j = 1, 2$. Each firm has one vacant job to fill. When filled, the job at firm j produces output z_j. Throughout the paper, I assume $z_j < \infty$. All agents have linear utility. If firm j hires a worker at wage w, the firm will receive a payoff of $z_j - w$ and the hired worker will receive a payoff of w. Because of search friction, there might be unmatched workers and unmatched firms at the same time. Workers that fail to find a match will receive their outside option of b. Firms that fail to find a match will receive their outside option of 0. Throughout this paper, I assume $b < z_j$. This assumption ensures there are always gains from trade of matches at every firm.

Trades unfold in four stages. In the first stage, firms simultaneously announce their wages, taking as given the other firm's wage and understanding the probability of hiring associated with different wage announcements. In the second stage, workers choose the probability of applying to firms, knowing the wage announcements from the first stage. In the third stage, workers and firms are matched in a frictional process. Workers cannot coordinate where to search for jobs. A firm can receive zero, one, or two applicants at the same time. If no applicant shows up, the firm stays vacant. If one applicant shows up, the firm makes a job offer to the only applicant. If two applicants show up, the firm randomly selects one worker to make the job offer to. In the fourth stage, workers who get a job offer decide whether to accept or reject it.

This trading environment is otherwise identical to the one considered in Burdett et al. (2001), except for the cost of directing search. If workers apply to two firms with probability (q_1, q_2), they need to pay a cost proportional to the Kullback-Leibler divergence (hereafter, *K-L divergence*) of the chosen probability (q_1, q_2) from $(\frac{1}{2}, \frac{1}{2})$, a strategy that applies randomly to firms:

$$\text{Cost of Directing Search} = c\left(q_1 \log \frac{q_1}{1/2} + q_2 \log \frac{q_2}{1/2}\right).$$

Specifically, the cost of directing search is a per-unit cost of search c multiplied by the expected likelihood ratio between (q_1, q_2) and $(\frac{1}{2}, \frac{1}{2})$, evaluated using distribution (q_1, q_2)[3]. I first define the symmetric subgame perfect equilibrium:

[3] The qualitative results also hold for a general set of divergence measure called f-divergence, which I discuss in appendix C.

Definition 1 (Symmetric Subgame Perfect Equilibrium in the 2×2 Game)
A symmetric subgame perfect equilibrium is a tuple of $\{q^i_j(w_1, w_2), w^e_1, w^e_2\}$ *such that:*
1. (Firm) w^e_j *maximizes firm* j*'s profit, given* $q^i_j(w_1, w_2)$ *and* w^e_{-j}*.*
2. (Worker) $q^i_j(w_1, w_2)$ *maximizes worker* i*'s payoff, given* (w_1, w_2) *and* $q^{-i}_j(w_1, w_2)$*.*
3. (Symmetry) $q^1_j(w_1, w_2) = q^2_j(w_1, w_2)$ *for* $\forall w_1, w_2$*.*

I restrict attention to a symmetric equilibrium.[4] This equilibrium refinement is motivated by our goal of studying a large economy that involves many workers and firms. In the non-symmetric equilibria, workers avoid visiting the firm the other worker applies to with high probability, which requires strong coordination between workers. This assumption is unnatural in large economies. The symmetric equilibrium requires less information regarding the other worker's behavior, which is more applicable for large economies.

Comment on Setup.- In appendix E, I show the game with observed wages and the cost of directing search is equivalent to a game where workers do not observe the wages but can acquire information regarding wages. Here I provide a brief discussion of this equivalence. In the full model of information gathering, firms draw productivity from a common distribution and optimally decide on the wages to pay. Workers understand the structure of the game, but face the uncertainty regarding which firm is more productive than the other. They choose a conditional distribution of signals regarding wages, by paying a cost that is proportional to the reduction of entropy from the prior distribution to the posterior distribution of wages.

Two issues complicates the characterization. First, the choice space of workers is a conditional distribution, which is high-dimensional. However, because the action space is discrete (which firm to apply to) and workers have linear utility, all signal choice is behaviorally equivalent to a signal that directly recommends on which firm to apply to. Thus, paying a cost on information is equivalent to paying a cost on

[4] For a range of wage announcements (w_1, w_2), three subgame equilibria exist. Within these three equilibria, two equilibria involve workers concentrate application to a specific firm. Burdett et al. (2001) offers detailed discussion of this multiplicity. One equilibrium involves worker 1 applies to firm 1 with more probability and worker 2 applies to firm 2 with more probability. Another equilibrium involves the opposite pattern.

search strategy. Second, many equilibria arise depending on the assumption on off-equilibrium belief. I adopt the equilibrium refinement proposed by Ravid (2019): the equilibrium strategy of workers must be optimal for every wage, under some perturbation of firms' posting strategy. Intuitively, this refinement requires that the equilibrium information acquisition strategy optimally captures the deviation of wage postings from the firm side. It turns out for any realization of firm productivity, the equilibrium outcome in the full model is characterized by the subgame perfect equilibrium in the simpler setup in this section where workers observe the productivity and wage posting of both firms, and pay a cost on their strategy of search. [5]

Worker's Search Problem – Consider the search problem that worker i faces. She will take as given two objects: (i) the wage announcements from firms (w_1, w_2) and (ii) the search strategy of the other worker. Denote the strategy of worker i as (q_1, q_2) and the other worker's strategy as (q_1^{-i}, q_2^{-i}). Equation (1) lays out worker i's search problem:

$$\max_{q_1,q_2\in[0,1]} q_1 \left(1 - q_1^{-i} + \frac{q_1^{-i}}{2}\right) \max\{w_1 - b, 0\} + q_2 \left(1 - q_2^{-i} + \frac{q_2^{-i}}{2}\right) \max\{w_2 - b, 0\} \tag{1}$$

$$-c\left(q_1 \log \frac{q_1}{1/2} + q_2 \log \frac{q_2}{1/2}\right),$$

s.t.

$$q_1 + q_2 = 1.$$

Workers cannot coordinate their application, which generates search frictions. Therefore, worker i faces uncertainty about which firm worker $-i$ is applying to. In the event of worker i applying to firm 1, with $1 - q_1^{-i}$ probability the other worker does not apply to the same firm. In this case, she gets hired for certain. With probability q_1^{-i}, the other worker also applies to firm 1. In this case, the firm randomizes to hire either worker. As a result, worker i gets hired with probability $\frac{1}{2}$. Together, the probability

[5]The common productivity distribution assumption leads to using random search strategy as benchmark in calculating cost of directing search.

of getting hired conditional on applying to firm 1 is $1 - q_2^{-i} + \frac{q_1^{-i}}{2}$. Similarly, I can calculate the job finding probability if worker i applies to firm 2.

Conditional on getting hired, worker i gets an offer with wage w_j as promised. Worker i has the choice of walking away from the job offer, in which event she gets her outside option b. The offer-acceptance decision is simple: workers accept the job if $w_j > b$ and turn down the job offer if $w_j < b$. When $w_j = b$, workers are indifferent between accepting and rejecting the job, throughout the paper I assume workers take the job offer when they are indifferent.[6]

The net payoff of choosing strategy (q_1, q_2) is the difference in expected income and the cost of directing search. I assume workers apply for jobs for certain. This is a reasonable assumption, because to search is always strictly better than to stay out when at least one firm posts wage above workers' outside options.

The worker's problem is a strictly convex optimization problem. Solution (q_1^i, q_2^i) solving the first-order condition in equation (2) characterizes the unique optimal strategy:

$$c \log \frac{q_1^i}{q_2^i} = \left(1 - q_1^{-i} + \frac{q_1^{-i}}{2}\right)(w_1 - b)^+ - \left(1 - q_2^{-i} + \frac{q_2^{-i}}{2}\right)(w_2 - b)^+, \tag{2}$$

$$q_1^i + q_2^i = 1.$$

The optimal strategy balances the income-maximizing motive and cost-minimizing motive. It states that the marginal benefit of applying to the first firm must equal the marginal benefit of applying to the second firm at the optimum.

Here, I link the optimal search strategy to the notions of random search, directed search, and partially directed search. In the limit of $c \to 0$, workers will only apply to the firm with the highest expected payoff. In this case, search is directed. In the limit of $c \to \infty$, the optimal search decision is $q_1^i = q_2^i = \frac{1}{2}$. Worker i will not deviate from the random search strategy, because any deviation incurs a infinite cost. In this case, search is random. With $c \in (0, \infty)$, worker i applies to firm j with a higher probability if the expected payoff of applying to firm j is higher than that of firm

[6]This assumption ensures the existence of equilibrium, by making the payoff function of firms continuous when wage approaches b from the right.

$-j$. However, the solution to equation (2) is always in the interval $q_j^i \in (0,1)$. The optimal decision involves applying to both firms with positive probability, due to the convexity of the cost function. Related to the focus of this paper, search is partially directed for $c \in (0,\infty)$.

Subgame Equilibrium – I can now characterize the Nash equilibrium in the second stage: given any (w_1, w_2), worker i maximizes the payoff given worker $-i$'s equilibrium strategy, and vice versa.

Mathematically, the symmetric equilibrium simply requires that both workers use the same strategy: $q_j^1 = q_j^2 = q_j$. Taking the optimal search strategy from equation (2) and imposing the symmetric equilibrium, I reach the condition for a subgame equilibrium in equation (3):

$$c \log \frac{q_1}{q_2} = \left(q_2 + \frac{q_1}{2}\right)(w_1 - b)^+ - \left(q_1 + \frac{q_2}{2}\right)(w_2 - b)^+, \tag{3}$$

$$q_1 + q_2 = 1.$$

Lemma 1 states that there is a unique solution to the symmetric subgame equilibrium given any wage announcement (w_1, w_2). More importantly, conditional on the wage announcement, the firms' identities do not matter for the subgame equilibrium outcome. Based on this symmetry property, I can define the following function $q_j = Q(w_j, w_{-j})$ as the solution to equation (3).

Lemma 1 (Uniqueness of Subgame Equilibrium)

(1). Given any (w_1, w_2), there is a unique symmetric subgame equilibrium.

(2). The symmetric subgame equilibrium is independent of firm identity: given two wage announcements (w_1, w_2) and (w_1', w_2'), if $w_1 = w_2'$ and $w_2 = w_1'$,

$$q_1 = q_2', \quad q_2 = q_1'.$$

$Q(w_j, w_{-j})$ is the labor supply curve for firm j given firm $-j$'s announcement. When firm j announces a higher wage, workers will apply to firm j with a higher probability. The slope of this labor supply curve is governed by the cost of directing search c. Figure 1 plots the supply curve for different costs of directing search. I will refer to $Iq_j = IQ(w_j, w_{-j})$ the queue at firm j, which measures the expected number of workers waiting to get a job. Because I, the worker population, is a primitive of the model, I will use the terms queue and q_j, the probability of applying, interchangeably.

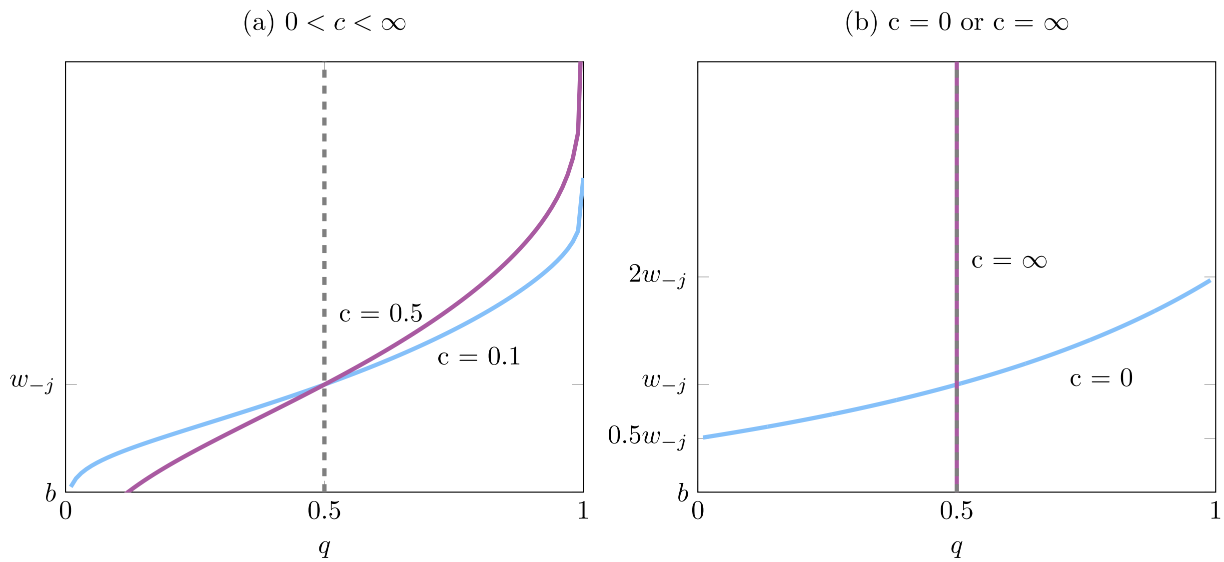

Figure 1: $Q(w_j, w_{-j})$ for different costs

Panel (a) plots the supply curve for positive and finite costs. Both labor supply curves are upward-sloping: to attract a longer queue, firm j has to post a higher wage. For all c, $Q(w_{-j}; w_{-j}) = \frac{1}{2}$: when firm j posts the same wage as the other firm, workers apply to firms with $(\frac{1}{2}, \frac{1}{2})$ regardless of the cost of directing search. When the cost of directing search increases, the labor supply curve becomes more inelastic. Because it is more costly to direct search, firm j must offer a higher wage than the low-cost case if it wants to attract the same queue length.

Panel (b) plots the supply curve for the two limiting cases: $c = 0$ or $c = \infty$. When the cost of directing search is zero, the labor supply curve is upward-sloping for a wage that is between half of the competitor's wage and twice the competitor's wage. This extreme case highlights the role of search friction. Given the competitor's wage posting, offering a slightly lower wage does not lead to zero application, because workers are also compensated by a higher job finding probability. However, when firm j posts a wage that is less than half of the competitor's wage, workers will apply to the competitor for sure ($q_j = 0$). When the cost of directing search is infinity, the labor supply curve is inelastic. Regardless of what wage firm j offers, workers never find it optimal to tilt their application strategy away from $(\frac{1}{2}, \frac{1}{2})$.

One property of this labor supply curve is crucial for our understanding of equilibrium: it has a flat segment when $w_j \leq b$. When the wage announcement falls below workers' outside option, workers regard firm j as equivalent to a firm that promised the outside option, because they will never take this job offer that is worse than their outside option. Firms will never post an unacceptable wage, because $w_j < b$ and $w_j = b$ will attract the same number of applicants but $w_j < b$ will result in zero hiring. Hereafter, I will only focus on wage that is weakly larger than b. This leads to a constraint $w \geq b$. Because it reflects workers' optimal job acceptance decision, I refer to this constraint as the participation constraint of workers.

The labor supply curve is upward-sloping, for three reasons. The first reason is the cost of directing search. For any $c \in (0, \infty)$, a higher queue requires a higher wage because workers need to be compensated for the cost of directing search to firm j; The second reason is the search friction. The third reason is the duopoly competition. We are in an economy where both firms internalize their impact on their competitors. In the equilibrium, they pay a lower wage because they understand competitors will respond in similar way. The upward-sloping labor supply curve implies that firms will pay workers below their productivity z_j in an equilibrium. This markdown on wage is due to (i) cost in directing search, (ii) the contribution of firms to the matching process, and (iii) size of the firms.

Subgame Perfect Equilibrium – Firms take as given the supply curve as well as their competitor's wage announcement. For a wage level w firm j announces, workers will apply to firm j with probability $q = Q(w, w_{-j})$, which is consistent with the subgame equilibrium given (w, w_{-j}). The probability of at least one worker apply is $1-(1-q)^2$. Conditional on hiring a worker, firm j receives a profit of $z_j - w$. Additionally, a firm faces the participation constraint $w \geq b$, because a wage offer below b will lead to zero hiring. Lemma 2 summarizes the discussion of subgame equilibrium so far. Proposition 1 establishes the existence and uniqueness of the equilibrium.

Lemma 2 (Subgame Perfect Equilibrium as Constrained Optimization)

$(w_1^e, w_2^e, q_1^e, q_2^e)$ is the outcome of a symmetric subgame perfect equilibrium if.f.:
1. w_j^e maximizes firm j's profit given w_{-j}^e:

$$w_j^e = \arg\max_w \; [1-(1-q)^2](z_j - w),$$

s.t.

$$q = Q(w; w_{-j}^e),$$

$$w \geq b.$$

2. (q_1^e, q_2^e) is the outcome of the subgame equilibrium given (w_1^e, w_2^e):

$$q_j^e = Q(w_j^e, w_{-j}^e).$$

Proposition 1 (Existence and Uniqueness)
For any (z_1, z_2, b, c), there exists a unique symmetric subgame perfect equilibrium.

I now investigate two cases of productivity combination. In the first case, both firms produce z when they match with workers. In the second case, $z_2 > z_1$. The first case highlights the effect of the cost of directing search on the equilibrium wage. The second case highlights the effect of cost in directing search on the allocation of workers across firms and the efficiency of market equilibrium.

Homogeneous Productivity: Monopsony – Suppose $z_1 = z_2 = z$. This case admits an analytical solution to the subgame perfect equilibrium. I start with a guess that both firms post wage w^* and attract $q^* = \frac{1}{2}$. With this guess, the best response of both firms collapses to a univariate equation in w^*. Solving this equation yields the result in Corollary 1.1.

Corollary 1.1 (Equilibrium Outcome with Homogeneous Productivity)
If $z_1 = z_2 = z$, the equilibrium outcome is

$$w_1^e = w_2^e = w^* = b + \max\left\{\frac{z-b}{2} - 2c, 0\right\},$$

$$q_1^e = q_2^e = q^* = \frac{1}{2}.$$

This simple analytical result offers us several economic insights. First, although in the equilibrium no cost is paid to direct search, the wages are reduced. A higher cost of directing search leads to a lower level of equilibrium wage. Because workers cannot perfectly target high-wage firms, firms find sharing output with workers is less attractive. Second, the equilibrium wage is a function of the split of gains from trade according to matching process $\frac{z-b}{2}$ minus markdown due to the cost of directing. Because the matching process I consider here is symmetric, in that workers and firms contribute to a successful match in the same way, both worker and firm contribute half to the matching process. The rent extracted by the firm is $2c$. It is the cost of directing search adjusted by the equilibrium probability of job finding $\frac{1}{2}$.

What happens when cost of directing search increases? When the cost of directing search increases, the labor supply curve becomes steeper given the old equilibrium wage w^e. One of the two firms will find it optimal to deviate to a lower wage and shorter queue. Given the deviating firm's choice of lower wage, the supply curve for non-deviating firm shifts down. This deviation argument can be applied step by step. Each step pushes down equilibrium wage, until $q = \frac{1}{2}$ is again the optimal queue for both firms. Worker applies with $(\frac{1}{2}, \frac{1}{2})$ in equilibrium, regardless of the cost. However, the wage is lower and firms are making higher profit. As the cost continues to rise, both firms will be bounded by the participation constraint of workers and post the outside option b.

Heterogeneous Productivity: Inefficiency – Now consider the case with $z_1 < z_2$. With this case, I ask two questions: First, how does the equilibrium allocation of workers depend on the productivities of two firms and cost of directing search? Second, how does the equilibrium allocation compares to the constrained efficient allocation?

Corollary 1.2

If $z_1 < z_2$*, then* $q_1^e \leq q_2^e$ *and* $w_1^e \leq w_2^e$*. The inequality is strict if* $\max\{w_1^e, w_2^e\} > b$*.*

In the subgame perfect equilibrium, the more productive firm posts a (weakly) higher wage than the unproductive firm. The opportunity cost of not hiring is higher for the productive firm, because a filled job produces more at this firm. Thus the productive firm is more willing to post a higher wage in order to attract applicants. As a result, the more productive firm attracts a longer queue in the equilibrium. The comparison is weakly because of the participation constraint $w \geq b$. When cost of directing search is high enough, firms will be bounded by the participation constraint.

Corollary 1.3

For any (z_1, z_2, b)*, there are two thresholds* $(\bar{c}_1, \bar{c}_2)$ *such that:*
For $c < \bar{c}_j$*,*

$$w_j^e(c) > b.$$

For $c \geq \bar{c}_j$*,*

$$w_j^e(c) = b.$$

Moreover, $\bar{c}_2 > \bar{c}_1$ *if* $z_2 > z_1$*.*

Corollary 1.3 shows there are two thresholds of the cost. Below the lower threshold, both firms post wages that are strictly above workers' outside option; between the lower and the higher threshold, the unproductive firm is bounded by participation constraint while the productive firm still posts wage strictly larger than b; above the higher threshold, both firms post workers' outside option.

Corollary 1.4

Denote $\{q_j^e(c), w_j^e(c)\}$ *the equilibrium with cost of directing search* c*. If* $c_2 > c_1$*, then* $|q_1^e(c_1) - q_2^e(c_1)| \geq |q_1^e(c_2) - q_2^e(c_2)|$*. The inequality is strict if* $\max\{w_1^e(c_1), w_2^e(c_1)\} > b$*.*

The cost of directing search dampens the difference of expected hiring between the two firms through two mechanisms. Mechanically, it is more costly for workers to differentiate the two firms when the cost is high, regardless of the wage. Moreover, the cost of directing search interacts with the participation constraint $w \geq b$. When the cost is high enough, the wage differential between two firms vanish, which further dampens the difference of expected hirings between two firms.

Is the wage equilibrium constrained efficient? Suppose there is a social planner who instructs how workers apply to these two firms to maximize the net production of the economy. The planner is subject to two types of frictions: (i) search friction – the planner has to instruct workers to apply with the same strategy; and (ii) cost of directing search – the planner has to pay the cost of directing search. With the assumption $z_j > b$, the planner will always instruct a match to form whenever possible. The planner's problem is as in equation (4):

$$\max_{q_1,q_2\in[0,1]} \sum_{j=1}^{2}(1-(1-q_j)^2)(z_j-b) - 2c\sum_{j=1}^{2} q_j \log\frac{q_j}{1/2}, \tag{4}$$

s.t.

$$q_1 + q_2 = 1.$$

The optimal allocation will equate the net social benefit of applying to firm j to the shadow value of workers (V^*). The efficient allocation can be found as the solution to the three-equation system in terms of (q_1^*, q_2^*, V^*) as in (5):

$$(1-q_1^*)(z_1-b) = c\log q_1^* + V^*, \tag{5}$$

$$(1-q_2^*)(z_2-b) = c\log q_2^* + V^*,$$

$$q_1^* + q_2^* = 1.$$

The equilibrium allocation is inefficient. In Figure 2, I plot the allocation from the subgame perfect equilibrium and the socially efficient allocation, for different cost of directing search and for different z_2, while holding $z_1 = 1$ and $b = 0$.[7] As the z_2 increases, the second firm becomes more productive. Both the equilibrium and the planner increase the queue to the second firm. However, for all finite cost (Panel (a)-(c)), there is a gap between the equilibrium allocation and the planner's allocation, in that the equilibrium assigns too many applicants to the unproductive firm than the efficient allocation.

This inefficiency comes from two forces. The first force is due to the duopoly competition in finite economy (Galenianos et al., 2011). In the equilibrium, both firms behave strategically. They internalize how their wage announcement affects the other firm. This is the inefficiency present in standard duopoly models. In the equilibrium, the productive firm does not hire enough compare to the efficient allocation and the unproductive firm hires too much. The second inefficiency is novel in partially directed search models. It comes from the binding constraint of wages at workers' outside options. In the equilibrium, both firms extract markdown from wages. The markdown

[7] This normalization is without loss of generality, because the allocation is homogeneous of degree zero in (z_1, z_2, b, c) and the productivity can always be relabelled as $z_j' = z_j - b$.

per se does not create inefficiency if both firms extract the same markdown. However, the markdown differs when the unproductive firm is constrained by the participation constraint of wages. When the participation constraint binds, the more productive firm extracts a higher markdown and hires less than the efficient allocation. In Figure 2, inefficiency due to binding participation constraint shows up as the flat segment of equilibrium allocation. Related to the result in Corollary 1.3, when z_2 is low enough, the unproductive firm does not post wage above b and the market assigns a constant q_2 for a range of low z_2.

The inefficiency due to the worker's participation constraint is non-monotonic in the cost of directing search. When the cost is zero, the only inefficiency is due to duopoly competition in the finite economy. When the cost is infinite, the constraint binds for both firms and results in a random search equilibrium. The planner's solution is also random search because the planner does not have any ability to distort search to different firms.

The inefficiency due to duopoly competition will vanish when the economy is large enough, when firms' impact on the market outcome is negligible. To isolate the inefficiency due to cost of directing search, I consider a limiting economy where both firms and workers' population grow to infinity. Section 2.2 prepares the discussion of limiting economy by consider an $I \times J$ economy, with $I, J \geq 2$.

2.2. *$I \times J$ Economy*

Setup. – The economy has I workers and J firms. Workers are indexed by $i = 1, ..., I$ and firms are indexed by $j = 1, ..., J$. Other than the population, this environment is identical to the case with two workers and two firms.

First, I modify the cost of directing search to accommodate more than two firms. The random search strategy is now $(\frac{1}{J}, ..., \frac{1}{J})$:

$$\text{Cost of Directing Search} = c \sum_{j=1}^{J} q_j \log \frac{q_j}{1/J}.$$

In the general case with more than two workers or firms, establishing uniqueness is challenging. In the 2×2 game, I can summarize the labor supply to a firm as function

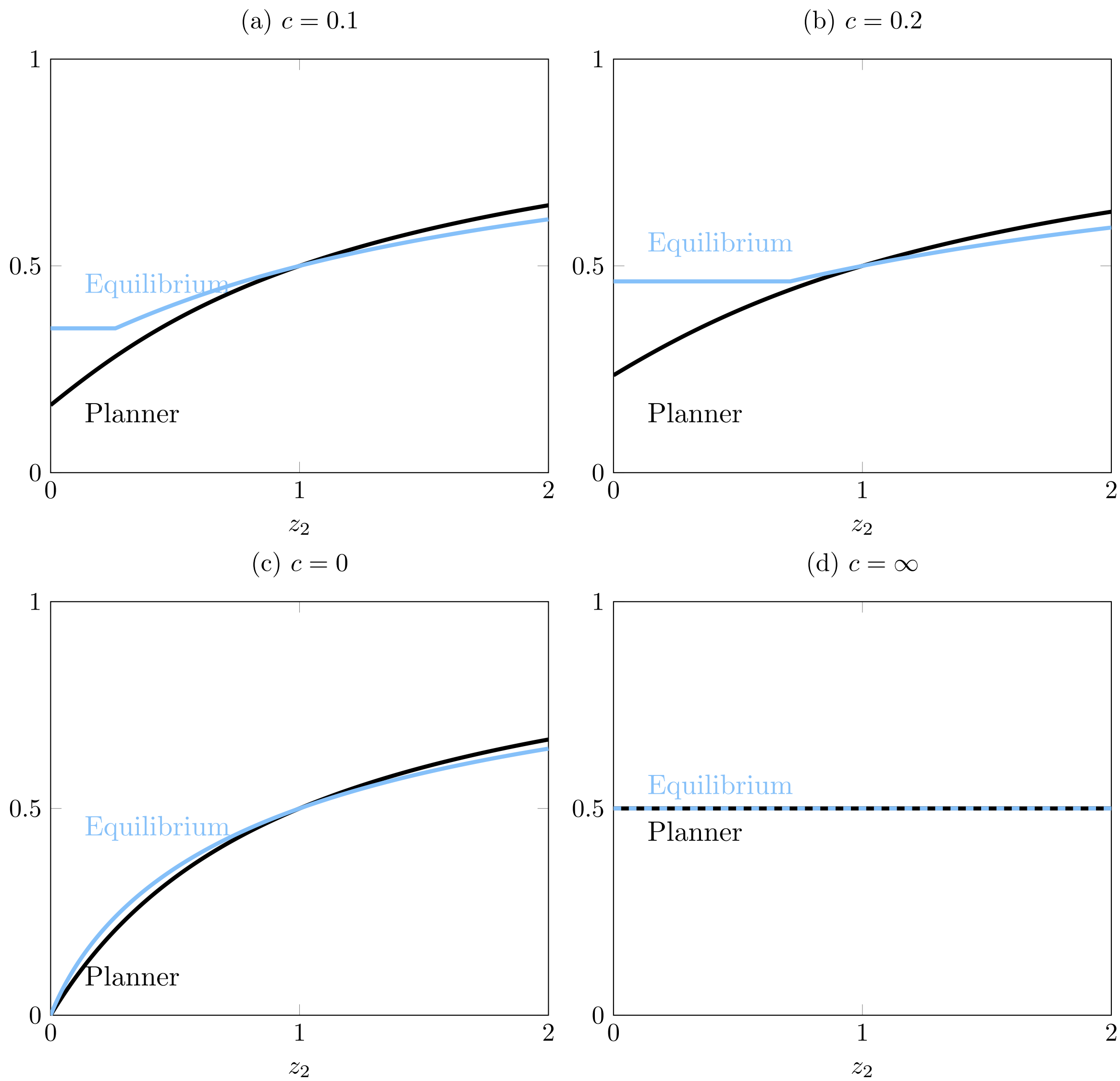

Figure 2: Equilibrium and Efficient Allocation (q_2)

Note: all plots normalize $z_1 = 1$ *and* $b = 0$.

of its own wage and the competitor's wage. In the $I \times J$case, this function depends on the vector of competitors' postings. I rely on the fixed-point theorems to establish the existence of subgame perfect equilibria. However, these statements normally do not guarantee uniqueness.

Equilibrium Conditions. – Because the economic intuition is the same as 2×2 case, I directly summarize the the key results in the $I \times J$ games. The details of the derivation can be found in the Appendix.

Proposition 2 (Existence of Symmetric Equilibrium)
At least one symmetric subgame perfect equilibrium exists.

Homogeneous Productivity– Corollary 2.1 characterizes the unique symmetric equilibrium when $z_j = z$. In this unique equilibrium, firms post the same wage w^* and workers apply to firms with equal probability:

Corollary 2.1
There is a unique symmetric equilibrium if productivity is the same for all firms $z_j = z$. In this equilibrium, workers apply to each firm with probability $\frac{1}{J}$. Let $\mu = \frac{I}{J}$ be the exogenous worker-firm ratio in the economy. The equilibrium wage is

$$w^* = b + \max\left\{\mu \frac{(1-\frac{1}{J})^{\mu J}(z-b) - c}{1-(1-\frac{1}{J})^{\mu J} - \frac{1}{J-1}(1-\frac{1}{J})^{\mu J}\mu}, 0\right\}.$$

The equilibrium wage converges when $J \to \infty$ and $I = \mu J$:

$$\lim_{J\to\infty} w^* = b + \max\left\{\mu \frac{e^{-\mu}}{1-e^{-\mu}}(z-b) - \mu \frac{1}{1-e^{-\mu}}c, \quad 0\right\}.$$

2.3. *Lessons from the Finite Economy*

In this section, I characterized a model of partially directed search in the finite economy. I highlight two takeaways on the mechanisms: (i) costly directed search leads to a new type of monopsony power and (ii) the cost of directing search leads to inefficiency. Next, I characterize the equilibrium in the limiting economy when the population of workers and the population of firms grow to infinity. The limiting economy allows us (i) to characterize the equilibrium wages and allocations in closed-form and (ii) to focus on the inefficiency due to costly directed search when the monopsony power due to the size of firms vanishes.

3. LIMITING ECONOMY: ENTROPIC COMPETITIVE SEARCH

Setup – The economy has measure μ of workers and measure 1 of firms. Workers are indexed by $i \in [0, \mu]$ and firms are indexed by $j \in [0, 1]$. Each firm has one vacant job to fill. When filled, the job at firm j produces output z_j.[8] All agents have linear utility. If firm j hires a worker with wage w, the firm will receive a payoff of $z_j - w$ and worker will receive a payoff of w. For workers that fail to find a match, they receive the outside option of b. For firms that fail to find a match, they receive the outside option of 0. Without loss of generality, I assume $z_{j'} \geq z_j$ when $j' > j$.

To make the limiting economy more general to incorporate other frictions, the matching process is characterized by a generalized matching function $n(q)$, where q is the queue length at a firm and $n(q)$ is the probability of this firm meeting a worker. For workers applying to this very firm, their probability of meeting the firm is $m(q) = qn(q)$, with both $m(q)$ and $n(q)$ differentiable. First, I assume $n'(q) > 0$ and $m'(q) < 0$. When there are more workers applying to the same firm, the firm have a higher chance of meeting workers and workers have a lesser chance of finding a job. Addtionally, I assume $n''(q) < 0$ and $m''(q) > 0$. When there are more workers applying, the marginal return of job filling probability to applicants declines. This concavity assumption captures the congestion in the labor search process. Additionally I define $\epsilon(q) = \frac{n'(q)}{m(q)}$ to be the elasticity of job filling probability to queue. This elasticity measures the contribution of a marginal worker to the matching process. From the assumption on n and m, $\epsilon(q)$ is decreasing in q: The contribution of a marginal applicant to a match is smaller when there are more workers applying to the same firm. This generalized matching process nests the matching process in the finite game as a special case. This special case is referred to as the urn-ball matching process in the search literature.[9] I return to the urn-ball process in the discussion of convergence from finite games to

[8] As a slight abuse of notation, I will use subscript to denote mapping from firm identity to outcomes to be consistent with the finite economy

[9] More specifically, when there are I workers that apply to a firm with probability Q, the probability of this firm meets a worker is $1 - (1 - Q)^I$ and the probability that a worker gets an offer from this firm is $\frac{1-(1-Q)^I}{IQ}$. As $I \to \infty$ holding $IQ = q$, these two probabilities limit to $n(q) = 1 - e^{-q}$ and $m(q) = \frac{1-e^{-q}}{q}$

limiting equilibrium.

The wage posting game unfolds in the same order as the finite economy. Here, I note the difference in the limiting economy. Because I am now defining an economy with a continuum of firms, I need to adapt the definition of search strategy and the cost of directing search. Workers' search strategy is a CDF on the interval of $[0,1]$. Define this CDF as A_j. The cost of directing search is the K-L divergence between chosen search strategy A_j and the random search strategy on the continuum of firms. Formally, it is calculated based on the Radon-Nikodym derivative of A_j with respect to the uniform distribution on $[0,1]$.[10] For the Radon-Nikodym derivative to be well-defined, A_j needs to be absolutely continuous with respect to the uniform distribution (the Lebesgue Measure on $[0,1]$), meaning it has the probability density function a_j on $[0,1]$. I will restrict attention to continuous distribution and use the following definition for cost of directing search:

$$\text{Cost of Directing Search} = c\int_0^1 a_j \log a_j dj.$$

I made this restriction to maintain mathematical coherence. Economically, it is not a very restrictive assumption, if I approximate the degenerate distribution as the limit of continuous distributions with shrinking supports. Although I cannot directly define the K-L divergence between a discrete distribution and a continuous distribution, I can take the limit of the cost associated with the sequence of continuous distributions as the cost for degenerate distribution. The K-L divergence asymptotes to infinity when the support shrinks. For this reason, I will exclude degenerate distributions from the choice set of workers.

3.1. *Subgame Equilibrium*

As a primitive step, I first analyze the subgame given any wage profile w. Worker's problem is as in equation 6. Workers take as given the wage profile $w : [0,1] \mapsto [b, \max_j z_j]$ and choose the probability density function of applying to firm j, to max-

[10]Radon-Nikodym derivative is f_j such that $A_J = \int_J f_j dV_j$ for any $J \subset \Omega([0,1])$

imize their payoffs:

$$a^e = \arg\max_a \int_0^1 m(q_j) \max\{w_j - b, 0\} a_j dj - c \int_0^1 a_j \log a_j dj, \tag{6}$$

s.t.

$$\int_0^1 a_j dj = 1.$$

The optimal search decision of the workers is characterized by the first-order condition as in equation 7:

$$m(q_j) \max\{w_j - b, 0\} - c \log a_j = V. \tag{7}$$

Workers equalize the net benefit of applying to every firm to a constant V. This constant is the multiplier for the constraint $\int_0^1 a_j dj = 1$ adding a constant c. With K-L divergence, V is also the expected net payoff before workers send out any applications.[11] From here on, I will refer to V as the market utility of workers, or the expected value of search.

Similar to the subgame perfect notion in the finite games, the workers' optimal search decision regulates the mapping from wage to queues for the firms. To derive this mapping, I impose $q_j = \mu a_j$ based on the symmetric equilibrium refinement. The queue at firm j equals the exogenous measure of workers and the probability that each of them applies to firm j. Equation 8 defines the subgame equilibrium mapping from wage to queue:

$$m(q) \max\{w - b, 0\} - c \log \frac{q}{\mu} = V. \tag{8}$$

To mimic the subgame perfect equilibrium in the finite economies, I further require equation 8 to hold for all $w \in [b, \max_j z_j]$, even for the off-equilibrium wages. The solution to equation 8 is unique for every w, given a fixed market utility V. Define this solution as $Q(w; V)$. This is the labor supply curve in the limiting economy.

[11]To see this: Integrate the first-order condition in 7 with weight a_j I get: $V = \int_0^1 m(q_j) \max\{w_j - b, 0\} a_j dj - c \int_0^1 a_j \log a_j dj$.

3.2. *Equilibrium Definition*

The equilibrium in the limiting economy inherits the spirit of competitive search models, in that firms take the wage-queue mapping in equilibrium as given to maximize their payoffs. Yet it differs in the assumption on how much information is available to workers.[12] Because the cost of directing search is rooted in Shannon's entropy, I refer to this new equilibrium concept as the *entropic competitive search equilibrium.* The entropic competitive search equilibrium is a tuple $\{w^e, q^e, V^e\}$, where w_j^e is the wage posted by firm j, q_j^e is the equilibrium queue at firm j, and V^e is the market utility of workers.

Definition 2 gives the conditions for an entropic competitive search equilibrium. Condition (i) requires that w_j^e maximizes the profit of firm j given the labor supply curve as defined in equation 8 and the market utility of workers V^e. Condition (ii) requires that q_j^e is consistent with the subgame equilibrium given w^e where all workers maximize their payoff and behave symmetrically. Condition (iii) requires that given the market utility V^e and the corresponding queue q^e, the total measure of applicants equals the exogenous measure of workers. The competitive nature of this equilibrium is that the firms take the market utility V^e as given[13]. However, they have monopsony power: A higher wage does lead to a longer queue. This monopsony power is summarized by the elasticity of the labor supply curve $Q(w; V^e)$.

Definition 2 (Entropic Competitive Search Equilibrium)
An entropic competitive search equilibrium is $\{w^e, q^e, V^e\}$ *such that the following conditions hold:*

(i). (optimal posting) w_j^e *solves firm j's profit maximization problem given* $Q(w; V^e)$*:*

$$w_j^e = \underset{w \in [b, z_j]}{\arg\max}\, n\bigg(Q(w; V^e)\bigg)(z_j - w),$$

(ii). (optimal search) q^e *is consistent with the subgame equilibrium given* w^e

$$q_j^e = Q(w_j^e; V^e),$$

[12] In competitive search equilibrium, workers have full information.

[13] A more precise definition of this market utility is the shadow value of search opportunity.

(iii). (market clearing) the total measure of queue equal the exogenous measure of workers:

$$\int_0^1 q_j^e dj = \mu,$$

For all $w \in [b, \max_j z_j]$, $Q(w; V^e)$ *is the solution to the following equation:*

$$m(q) \max\{w - b, 0\} - c \log \frac{q}{\mu} = V^e.$$

3.3. *Equilibrium Characterization*

First consider firm j's problem given the equilibrium market utility V^e. Firm j faces a constrained optimization problem as in equation (9):

$$\max_w \quad n(q)(z_j - w), \tag{9}$$

s.t.

$$q = Q(w; V^e),$$

$$w \geq b.$$

The profit of firm j is the probability of hiring $n(q)$ times the profit per worker $z_j - w$. The feasible combinations of (q, w) must be consistent with the labor supply curve given the equilibrium market utility V^e. Additionally, the firm faces the participation constraint on wage $w \geq b$.

Instead of writing the problem is terms of the wage w, I rewrite the problem in terms of queue q_j. Utilizing the property of the matching function $n(q) = qm(q)$ and the definition of labor supply curve $Q(w; V^e)$, the firm's problem in terms of queue is:

$$\max_q n(q)(z_j - b) - q(V^e + c \log \frac{q}{\mu}), \tag{10}$$

s.t.

$$V^e + c \log \frac{q}{\mu} \geq 0.$$

Equation 10 allows us to interpret the entropic competitive search equilibrium in a different way. Firm j makes the optimal input decision to maximize its profit.

The input is queue, the number of applicants. Inputting q units of queue leads to output of $n(q)(z_j - b)$. The cost of production has two components: (i) a market price for applicants V^e and (ii) a convex cost of increasing the input $c \log \frac{q}{\mu}$. These two components highlights the competitive nature and the monopsonic nature of the entropic competitive search equilibrium. Firm j's problem has a strictly concave profit function and a convex choice set. There is a unique q_j^e that maximizes firm j's profit, as in the first-order condition:

$$\underbrace{n'(q_j^e)(z_j - b)}_{\text{MPL}} - \underbrace{c(1 - \frac{\gamma_j}{q_j})}_{\text{Markdown}} = \underbrace{V^e + c \log \frac{q_j^e}{\mu}}_{\text{ACL}}, \tag{11}$$

with complementary slackness:

$$\gamma_j \left(V^e + c \log \frac{q_j^e}{\mu} \right) = 0,$$
$$\gamma_j \geq 0,$$
$$V^e + c \log \frac{q_j^e}{\mu} \geq 0.$$

The marginal value of an additional applicant is $n'(q_j^e)(z_j - b)$, the marginal increase of job filling probability multiplied by the gain from trade. The average cost of one applicant is $V^e + c \log \frac{q_j^e}{\mu}$, the market utility plus the cost of directing search. Due to the cost of directing search, there is a markdown $c(1 - \frac{\gamma_j}{q_j})$, where γ_j is the multiplier on the workers' participation constraint. When the participation constraint is slack, firms post a wage strictly above workers' outside option. In this case, firm j extracts a constant markdown c. When the participation constraint is binding, firms post a wage that equal workers' outside option. In this case, firm j's markdown depends on its productivity.

Now I can characterize the equilibrium wage from firm j by inverting the labor supply curve. With the definition of labor supply curve in equation 8, The average cost of applicant is $m(q_j^e)(w_j^e - b)^+$. Combining this result with the optimal queue decision of firm j, I reach the following equation for equilibrium wage from firm j:

$$w_j^e = b + \max \left\{ \epsilon(q_j^e)(z_j - b) - \frac{c}{m(q_j^e)}, 0 \right\}. \tag{12}$$

Workers are first compensated for their contribution to the matching process. Their share from the gains from trade is $\epsilon(q_j^e)(z_j - b)$. Due to the cost of directing search, firm j also extracts markdown $\frac{c}{m(q_j^e)}$ in wage unit. The participation constraint requires that the wage increment at firm j cannot be negative.

Next I establish the existence and uniqueness of the entropic competitive search equilibrium. The existence of equilibrium in this market relies on: (i) the continuity of the firm's optimal input in the market price; (ii) when the market price is 0, all firms demand a queue that is weakly larger than μ; (iii) When the market price is the highest productivity in the market, all firms demand a queue that is weakly smaller than μ. The uniqueness of equilibrium is based on the law of demand for applicants. From equation 11, the optimal input decision is weakly increasing in V^e, and strictly increasing in V^e if the participation constraint $w \geq b$ is not binding. As the market utility increases, the marginal applicant becomes more expensive. Firms respond by attracting less applicants. Because firms' profit maximization problem is concave, the optimal queue decision q_j^e is continuous in the market utility V. The aggregate measure of applicants follows the law of demand: When market utility increases, the aggregate queue strictly decreases. Proposition 3 formally states the existence and uniqueness of the entropic competitive search equilibrium.

Proposition 3 (Existence and Uniqueness of ECSE)
There is a unique entropic competitive search equilibrium.

How do equilibrium queues and wages depend on the productivities of firms? Equation 11 and equation 12 imply that more productive firms post higher wages and attract longer queues. All firms face the same upward-sloping labor supply curve. For the more productive firms, a marginal applicant is more valuable given the same level of queue. Therefore, the profit-maximizing queue for more productive firms must be higher than less productive firms. In order to attract a longer queue, more productive firms must promise workers higher levels of expected payoffs of applying. Workers are compensated by the job finding probability and the wage. As queues are longer at more productive firms, job finding probabilities are lower at these firms. In order to attract a longer queue, more productive firms must promise higher wages.

Corollary 3.1 (Productivity and Equilibrium Outcome)
Let $w(z)$ and $q(z)$ be the equilibrium wage and queue given productivity z. if $z > z'$:

$$w(z) \geq w(z')$$

$$q(z) \geq q(z')$$

with strict inequality if $w(z) > b$.

PROOF: The discussion so far already established the statement. *Q.E.D.*

How do equilibrium queues and wages depend on the cost of directing search? Corollary 3.2 establishes an important result for our understanding of the equilibrium outcome. Fix the the distribution of productivity in the economy, for every firm j, there is a threshold of cost of directing search, above which $w_j = b$ and firm j extracts a markdown that is below the unconstrained constant markdown c in the equilibrium. This threshold is increasing in firm j's productivity. This is a novel result: costly directed search interacts with the offer acceptance decision of workers, which generates endogenous dispersion of markdowns across firms. This is a similar result to 1.3, rephrased in the limiting economy.

Corollary 3.2 (Cost of Directing Search and Equilibrium Outcome)
Assume $n(q)$ has Inada condition: $\lim_{q\to 0} n'(q) = \infty$ and $\lim_{q\to\infty} n'(q) = 0$. For a given productivity distribution z_j, denote $\{q(z;c), \gamma(z;c), w(z;c)\}$ the equilibrium queue, multiplier to the participation constraint, and wage for a firm with productivity z in an equilibrium with cost c. For every $z < \infty$, there is $0 < \bar{c}(z) < \infty$ such that: if $c \geq \bar{c}(z)$,

$$\gamma(z;c) \geq 0,$$

$$w(z;c) = b;$$

if $c < \bar{c}(z)$,

$$\gamma(z;c) = 0,$$

$$w(z;c) > b.$$

Moreover, $\bar{c}(z_2) > \bar{c}(z_1)$ if $z_2 > z_1$.

3.4. *Convergence of Finite Games to Entropic Competitive Search Equilibrium*

This section establishes the convergence of the subgame perfect equilibrium in wage posting games with finite population to entropic competitive search, when the number of firms grows to infinity and the number of workers grows proportionally to firm population. The finite game is based on the urn-ball matching process. This matching process converges to a matching function of $n(q) = 1 - e^{-q}$. The plan of this section is to show that as the population of workers and firms grows to infinity, the subgame perfect equilibrium outcome converges to the outcome of the entropic competitive search equilibrium with matching function $n(q) = 1 - e^{-q}$ and the same productivity distribution.

Homogeneous Firms – From equation (11) and equation (12), in the entropic competitive search equilibrium with homogeneous firms, the equilibrium queue length is $q_j = \mu$ and the equilibrium wage is $b + \max\left\{\mu\frac{e^{-\mu}}{1-e^{-\mu}}(z-b) - \mu\frac{1}{1-e^{-\mu}}c, 0\right\}$. This wage is exactly the limit of subgame perfect equilibrium outcome from corollary 2.1, when the populations grow to infinity. Thus the outcome of subgame perfect equilibrium with homogeneous firms indeed converges to the outcome of entropic competitive search equilibrium with homogeneous firms.

Heterogeneous Firms – To make the notion of convergence explicit, consider a sequence of finite economies with growing populations and the same productivity distribution. Start with any finite economy economy with I workers, J firms, outside option b, and productivity $\mathbf{z} = (z_1, ..., z_J)$. For any positive integer t, define the $t-$replica economy as an finite economy with tI workers, tJ firms, outside option b, and productivity $\mathbf{z}^t = \cup^t(z_1, ..., z_J)$. As shown in proposition 2, there is at least one symmetric subgame perfect equilibrium. Denote (one of) the equilibrium allocations and wages in the $t-$replica economy as $(\mathbf{q}^t, \mathbf{w}^t)$. Correspondingly, we can derive the unique entropic competitive search equilibrium in the limiting economy with measure $\mu = \frac{I}{J}$ workers, measure 1 of firms, and the productivity distribution according to $\mathbf{z}^t = (z_1, ..., z_J)$. Define the equilibrium queue and wage as $(\mathbf{q}^\infty, \mathbf{w}^\infty)$, where (q_j^∞, w_j^∞) is the equilibrium outcome for a firm with productivity z_j in the limiting equilibrium.

Economically, by convergence, I mean the equilibrium queue and wage of a firm with productivity z_j limits to the outcome of an identical firm in the limiting equilibrium. Mathematically, for the sequence of finite game outcomes $\{\mathbf{q}^t, \mathbf{w}^t\}_t$, there is at least one subsequence $\{\mathbf{q}^{t'}, \mathbf{w}^{t'}\}_{t'}$ such that as $t' \to \infty$, for every j:

$$w_j^{t'} \to w_j^{\infty},$$

$$tIq_j^{t'} \to q_j^{\infty}.$$

Because $(\mathbf{q}^t, \mathbf{w}^t)$ is the outcome of the symmetric subgame perfect equilibrium in the t-replica economy, the marginal benefit of applying to every firm must equal. Therefore, we can find a unique market utility V_t such that for $\forall j$:

$$V_t = \frac{1 - (1 - q_j^t)^{tI}}{tIq_j^t}(w_j - b)^+ - c \log \frac{q_j^t}{1/(tJ)}.$$

Lemma 3 is the key step to establish convergence. If the market utility V_t converges to a limit V_∞, then the optimal choice of queue and wage for a firm with productivity z_j must also converges to the solution to the firm's problem with the same productivity z_j, given market utility V_∞ in the limiting equilibrium. This result is an application of the Maximum Theorem. Firm's problem is strictly concave. Therefore, the optimal posting decision in the finite games is continuous in its parameters $(t, V_t, \mathbf{q}^t, \mathbf{w}^t)$. The limit of these optimal solutions is the solution to the first-order condition when t goes to infinity. In the limit, the impact of any marginal firm on market utility V_t vanishes, and the matching probabilities limit to the urn-ball matching function.

Lemma 3 (Convergence of Firm's Optimization Problem)
If $V_t \to V_\infty$ as $t \to \infty$, then (w_j^t, tIq_j^t) converges to $(\tilde{w}_j, \tilde{q}_j)$ such that

$$(\tilde{q}_j, \tilde{w}_j) = \arg\max_{q,w} (1 - e^{-q})(z_j - w),$$

s.t.

$$\frac{1 - e^{-q}}{q}(w - b)^+ - c \log \frac{q}{\mu} = V_\infty.$$

Proposition 4 is a natural extension of convergence in Lemma 3. Proposition 4 shows that (i) there exists a subsequence of the replica economies whose equilibrium outcomes converges and (ii) every convergent subsequence must has the outcome of the entropic competitive search equilibrium as limit. I make statement only on subsequences of the replica economies because multiple subgame equilibria might exist for the same firm-worker population and productivity distribution. First, the sequence of equilibrium market utility $\{V_t\}_t$ is a bounded sequence, because the equilibrium wages cannot exceed the productivities of firms and worker's optimization will never make the market utility negative infinity. I can always find a convergent subsequence $\{V_{t'}\}_{t'}$. For this convergent subsequence, the conditions for Lemma 3 is satisfied. The only step is to show that the limit V_∞ must also clears the market for applicants in the corresponding limiting economy, because of the continuity of firm strategies in market utility V_t. This argument establishes the existence of convergent subsequence of equilibrium outcomes. Second, the entropic competitive search equilibrium is unique given any productivity distribution. Therefore, every convergent subsequence of the replica economies must has the outcome of this unique entropic competitive search equilibrium as limit. Conversely, for every entropic competitive search equilibrium with discrete productivity distribution, we can find a sequence of replica economy whose outcomes converge to the equilibrium of this entropic competitive search equilibrium.

Proposition 4 (Convergence of Equilibrium with Discrete Distributions)

1. For a sequence of the symmetric subgame perfect equilibrium outcomes $\{\mathbf{w}^t, \mathbf{q}^t\}_t$*, there exists at least one subsequence* $\{\mathbf{w}^{t'}, \mathbf{q}^{t'}\}_{t'}$ *such that for* $\forall j$*:*

$$w_j^{t'} \to w_j^\infty,$$

$$t' I q_j^{t'} \to q_j^\infty.$$

2. For every entropic competitive search equilibrium with discrete productivity distribution, we can find a sequence of finite economies with the same productivity distribution such that the equilibrium converges.

The discussion so far assumes the share of productivity stays constant when I take population to infinity. This result offers us an interpretation of entropic competitive search equilibrium with discrete distribution of productivity. The following result shows if a sequence of productivity distribution converges weakly to a limit, then the associated equilibrium outcomes in distribution also converges to the equilibrium with limiting distribution. So an entropic competitive search equilibrium with continuous distribution can be interpreted as the limit of equilibria with discrete distributions. Intuitively, this result comes from two features of the limiting equilibrium: First, fact firms only interact with the market through market utility V; Second, the optimal posting solution is continuous in the market utility. If the market utility converges, the solution of firm's problem also converges. The market clearing condition in equivalent problem implies the market utility will converge in this case, which justifies the assumption.

Proposition 5 (Convergence of ECSE in productivity distribution)
Fix (b, μ), $z_j^n \xrightarrow{d} z_j^*$, *and* (q_j^n, w_j^n) *is the associated equilibrium outcome given* z_j^n. *Then* $q_j^n \xrightarrow{d} q_j^*$ *and* $w_j^n \xrightarrow{d} w_j^*$, *where* (q_j^*, w_j^*) *is the associated equilibrium outcome with* z_j^*

4. EFFICIENCY

I have characterized the allocations and wages in the unique limiting equilibrium. To understand the implications of the monopsony due to costly directed search on policies such as minimum wage, it is important to understand the efficiency property of the market equilibrium. Because the equilibrium allocation only depends on $z_j - b$, from here on, I will normalize $b = 0$ to simplify the notations.

4.1. *Efficient Allocation*

The planner instructs workers to apply to firms with distribution a_j. Due to lack of coordination, the planner needs to instruct all workers to apply with the same strategy.[14] The net output of the economy equals the sum of outputs from different firms minus the cost of directing search of all workers. The first constraint requires that the search strategy the social planner picks has to be an appropriately defined distribution. The second constraint states that the queue length at firm j is formed as the measure of workers in the economy times the probability of workers applying to a specific firm j.

[14] For example, planner cannot instruct half of workers to apply to firms $[0, 0.5]$ and the other half to $[0.5, 1]$.

$$\max_{a_j} \int_0^1 n(q_j) z_j dj - c\mu \int_0^1 a_j \log a_j dj,$$

s.t.

$$\int_0^1 a_j dj = 1,$$

$$q_j = \mu a_j.$$

The first-order condition on the queue length at firm j characterizes the unique optimal allocation. Denote the solution to this planner's solution as q_j^*. It has to solve equation system (13):

$$n'(q_j^*) z_j - c \log \frac{q_j^*}{\mu} = V^*, \tag{13}$$

$$\int_0^1 q_j^* dj = \mu.$$

When the marginal worker applies to firm j, the probability of a match for firm j increases by $n'(q_j^*)$. However, distorting search away fro firm distribution is costly. $c \log \frac{q_j^*}{\mu}$ measures the marginal cost of directing search. The socially optimal allocation must equal the net benefit of applying to firm j to a constant social value V^*.

4.2. *Efficiency Property of Equilibrium*

I compare the allocation from the entropic competitive search equilibrium to the constrained efficient outcome. Equation (14) summarizes the allocation from the entropic competitive search equilibrium:

$$n'(q_j^e) z_j - c \log \frac{q_j^e}{\mu} - (c - \frac{\gamma_j}{q_j}) = V^e, \tag{14}$$

$$\int_0^1 q_j^e dj = \mu.$$

Compared to the social planner's calculation, firms extract a constant markdown c in expectation, the markdown due to cost of directing search, from each applicant. γ_j is the multiplier on the participation constraint of workers $w_j \geq b$. When this constraint binds, γ_j is positive. Otherwise, $\gamma_j = 0$.

By comparing the allocation of the entropic competitive search equilibrium to the allocation of the planner's solution, I reach the following welfare theorems of partially directed search environment in proposition 6.

Proposition 6 (Welfare Theorems of Partially Directed Search Model)
The entropic competitive search equilibrium is constrained efficient if and only if:

$$\gamma_j = \gamma, \ \ for \ \forall j$$

The market equilibrium is efficient in four cases, depending on the value of c.

Case 1: random search. – In this case, both the equilibrium allocation and the socially efficient allocation have workers apply to every firm with equal probability, regardless of their productivities. In the random search equilibrium, all firms pay workers their outside options, and the participation constraint binds with $\gamma_j = \infty$.

Case 2: directed search. – In this case, the markdown due to costly directed search is zero and the participation constraint does not bind for any firm. Thus the equilibrium allocation and the efficient allocation coincide. Related to the statement in proposition 6, $\gamma_j = 0$.

Case 3: partially directed search with homogeneous firms. – In this case, both the equilibrium allocation and the socially efficient allocation have workers apply to every firm with equal probability, because they are the same. No matter the participation binds or not, it is the same across firms, $\gamma_j = \gamma$.

Case 4: partially directed search with non-binding participation constraint. – When the cost is low relative to firms' productivities, all firms promise wages strictly above workers' outside option and $\gamma_j = 0$. For any solution to the social planner's problem, relabel $V^e = V^* - c$. Given this V^e, the equilibrium queues are identical to the planner's solution and the market clears for applicants. Because the solution to planner's problem and the equilibrium outcome are both unique, I showed the social planner's solution is identical to the allocation from the entropic competitive search equilibrium.

Inefficiency arises under four conditions: (1) the cost of directing search is positive and finite; (2) firms are different in their productivities; (3) the participation constraint binds for a positive measure of firms. The participation constraint of workers forces unproductive firms to extract less markdown, because their unconstrained optimal wage is below workers' outside option. (If they can, they want workers to pay for a match.) As a result, the markdown is unevenly distributed among firms. Markdown at firm j decreases the payoff for workers to apply to firm j. When the markdowns at the productive firms are lower than the markdowns at the productive firms, the incentive of applying to different firms is distorted. Productive firms have more markdown and attract fewer workers than socially optimal; Unproductive firms have less markdown and attract more workers than socially optimal. As a result, the market equilibrium is inefficient. The discussion of efficiency highlights the following message: uneven markdown leads to distortion. The partially directed search environment endogenously generates this type of inefficiency, when the unproductive firms are forced to extract lower markdowns. The effect of cost of directed search on efficiency is non-monotonic: the inefficiency peaks at an intermediate level of cost.

In traditional monopsony models of the labor market (Robinson, 1969), market power leads to inefficiency. Here, I find a case where market power leads to pure rent extraction: it only affects the rent sharing between workers and firms, not the allocation efficiency. Why? In the traditional models of monopsony, firms face an upward-sloping labor supply curve because they are mutiworker firms with wage-setting power. To hire one more worker, they raise the market wage and have to pay all the existing workers within the firm a higher wage. The optimal choice of a monopsonistic firm is to ration on employment. They hire below the competitive level to avoid inflating wages within incumbent workers. The wedge between the socially efficient allocation and the market equilibrium with monopsony is on the margin of work or leisure in the traditional models.

In the model of costly directed search, firms face upward-sloping supply curves for a different reason: the cost of directing search weakens competition across firms. Firms set wages to attract workers. When the cost of directing search is high, the return to setting a higher wage decreases. Firms do extract markdown due to the costly directed search. However, this markdown does not lead to distortion when every firm extracts the same markdown. This result comes from the general equilibrium force. Conditional on the same market price for applicants (V), the upward-sloping supply curve leads to a lower level of queue length and lower wage. However, a lower wage length from every firm decreases workers' market utility from the market. The drop in the market utility makes it less costly for each firm to attract more applicants. It turns out with the specific cost function of K-L divergence, the underemployment and the general equilibrium forces cancel out. The potential wedge between the planner's solution and the market equilibrium is on the margin of which firm to work for. If all firms extract the same markdown, workers' search decision is not distorted by monopsony.

4.3. *Policy Implications*

With the welfare theorem in hand, I can analyze the implications for policies in a clear way. To recap, the efficiency of an entropic competitive search equilibrium depends on whether markdowns are equalized across firms. Inefficiency of the market equilibrium is due to unproductive firms being bounded by participation constraint $w \geq b$ and attracting more applicants than socially optimal.

Minimum Wage. – Suppose there is a minimum wage $\underline{w} \in (\min_j w_j^e, \min_j z_j)$. Given a minimum wage in this range, some firms need to increase their wages. However, as $\underline{w} < \min_j z_j$, all firms are still making positive profits in equilibrium, so they stay active. This restriction helps isolate the reallocation effect of minimum wage in a partially directed search environment, by assuming away the entry margin.

The only difference between the case with a binding minimum wage and the baseline environment is that firms now face a tighter constraint on the wage to $w \geq \underline{w} > b$. Otherwise, the firm's profit maximization problem is identical to the baseline model:

$$\max_{w,q} \quad n(q)[z_j - w],$$

s.t.

$$m(q)w - c \log \frac{q}{\mu} = V^e,$$

$$w \geq \underline{w}.$$

In an equilibrium with a binding minimum wage, firms are divided into two groups according to their productivities. The first group of firms are less productive. The unconstrained optimal wage for these firms is below the minimum wage. Because these wages are not feasible, they post the minimum wage and attracts the corresponding queue $\underline{q}$. The second group of firms are more productive. They are not constrained by minimum wage. The two types of firms are separated by a threshold productivity $\bar{z}$, with which the firm's unconstrained optimal wage is exactly the minimum wage. Firms with $z < \bar{z}$ are constrained by minimum wage and firms with $z \geq \bar{z}$ are unconstrained by minimum wage.

When the minimum wage increases, more firms become constrained. The threshold productivity $\bar{z}$ increases. Meanwhile, all firms below the new threshold are now forced to pay a higher wage. When the constrained firms post a higher wage, the market utility increases for workers. The unconstrained firms will also post higher wages, because they now face more competitions. As a result, posted wages increase for every firm after a minimum wage hike. Workers reallocate from the productive firms to the unproductive firms because the unproductive firms now extract a even lower markdown than the productive firms.

I then aggregate the impacts on firm-level outcomes to the impact on aggregate efficiency. Differentiating the planner's objective function with respect to the minimum wage I get the following response of welfare to the minimum wage, as in equation (15):

$$\mathbb{W}'(\underline{w}) = -\int_0^1 \gamma_j \frac{d \log q_j^e}{d\underline{w}} dj + V^e \int_0^1 \frac{dq_j^e}{d\underline{w}} dj = -\int_0^1 \gamma_j \frac{d \log q_j^e}{d\underline{w}} dj. \tag{15}$$

The first part of the equality is based on the optimal posting condition of firms. An increases in minimum wage tightens the constraint on wages $w \geq \underline{w}$ and also shifts the queue length at different firms.[15] The second equality comes from the market clearing for applicants. No matter how minimum wage shifts queues, it must aggregate up to zero. I reach a form of envelope condition for the welfare: the effect of minimum

[15]Here, I slightly abuse the notation because the queue at threshold is not differentiable with respect to minimum wage. The argument holds when will look at the left and right limit separately.

wage is the sum of the multiplier on the constraint $w \geq \underline{w}$. Equation 15 is the key result of this section: An increase in the binding minimum decreases efficiency of the equilibrium allocation, by creating more markdown dispersion across firms.

Note that minimum wage is a popular policy not only because of its pro-efficiency benefit in a standard monopsony. Many advocates for minimum wage stress its impact on redistribution from firms to workers. In the partially directed search models, minimum wage creates a new form of distortion: the minimum wage firms hire more often than is socially optimal. Policy-makers face a tradeoff between equity (to increase wages of workers) and efficiency.

Employment increases after a minimum wage hike, in line with the prediction of traditional monopsony models. However, this effect works through a very different mechanism. A higher minimum wage reallocates applicants from the productive firms to the unproductive firms. The productive firms have longer queue than the unproductive firms. Because of the search friction, the reallocation of workers alleviates search frictions in aggregate, which increase aggregate job finding probability. The average wage might increase or decrease after a minimum wage hike, although the posted wages increase at all firms. This result is due to the compositional effect. A minimum wage hike reallocates workers from high-wage firms to low-wage firms. The net effect on average wage depends on the comparison between the change in posted wages and the shift of worker allocation.

Profit Tax. – It is natural to ask whether an alternative policy instrument can alleviate the inefficiency while achieving the goal of redistribution. Suppose the corporate profit tax is $T(\pi)$. This tax does not change workers' search decision. Therefore, the firms in the economy still face the same labor supply curve. With the taxation, posting a wage w generates after-tax profit $z_j - w - T(z_j - w)$ for firm j. Equation 16 summarizes firm's problem with an arbitrary tax policy (I assume the tax function is well-behaved, and come back to verify):

$$\max_{w \geq 0, q} \quad n(q)\bigg(z_j - w - T(z_j - w)\bigg), \tag{16}$$

s.t.

$$m(q)w - c\log q = V,$$

$$w \geq 0.$$

The goal is to design the shape of the tax function $T(\pi)$ that will decentralize the social planner's problem while guaranteeing the workers are paid their social values. Proposition 7 states that there is a budget-balanced tax function that implements the planner's solution. Moreover, with this tax function, the equilibrium wage equals workers' contribution to the matching process. Therefore,the tax policy function also undoes the markdown due to the cost of directing search.

Proposition 7 (Optimal Corporate Profit Tax)

The following budget-balanced tax function implements the social planner's solution:

$$T'(\pi) = \frac{T(\pi) + \frac{c}{n'(q(\pi))}}{\pi + \frac{c}{n'(q(\pi))}},$$

$$\int_0^1 T(\pi_j^e)dj = 0,$$

where $q(\pi)$ solves

$$m(q)\frac{\epsilon(q)}{1-\epsilon(q)}\pi - c\log q = V^*.$$

Meanwhile workers are paid their social value:

$$w_j^e = \epsilon(q_j^*)z_j.$$

The marginal tax rage is increasing in π:

$$T''(\pi) > 0.$$

This transfer function starts as a subsidy and becomes a taxation when profit is high enough. To see this, notice the transfer is increasing when $T(\pi) > 0$. If the transfer function starts as a tax, the budget cannot be balanced. Therefore, the transfer function must starts as a subsidy for low-profit firms, $T(0) < 0$. The subsidy increases at first until it reach the threshold $T(\bar{\pi}) = -\frac{c}{n'(q(\bar{\pi}))}$. When profit is larger than this threshold, the marginal tax rate on profit becomes positive. The marginal tax rate is always below one. To see this, if there is some π with $T'(\pi) > 1$, there must be a $\tilde{\pi}$ such that $T(\pi') = \pi'$. This cannot happen because the transfer function starts at $T(0) < 0$ and $T'(\pi) < 1$ if $T(\pi) < \pi$. This transfer scheme is progressive, in that the marginal tax rate (or negative subsidy rate) is increasing in the profit of a firm. By making higher profit less attractive to the firm, the corporate profit transfer function incentivizes firms to post higher wage to workers.

The profit transfer policy redistributes from the productive firms to the unproductive firms and the workers. The productive firms are the ones that gain high profit in the equilibrium. By making extracting markdown less attractive, the transfer policy increase the posted wage at all firms. Unproductive firms are running lower profit due to the higher posted wage. The transfer policy then take the tax revenue from productive firms to subsidize the unproductive firms. By doing so, this policy makes all firms steer away from the participation constraint $w \geq b$ and equalizes the markdown across all firms to 0.

5. DISCUSSION

In this section, I discuss how the partially directed search model is linked to other applications, as well as the possible extensions to quantitative studies. Alternative parametrization of the cost function is also included in the appendix.

5.1. *Information Technology and the Labor Market*

The past decades have witnessed a rapid improvement of information technology. These improvements will affect how workers search for jobs, as well as the wages and the allocations in the labor market. Papers in the literature consider the improvement as changing the efficiency of matching function (e.g., Martellini and Menzio, 2018) or making more searchers informed (e.g., Lester, 2011). The partially directed search model provides an alternative way to interpret changing information technology: as information becomes cheaper to acquire, the cost of directing search falls. This section takes the baseline model to consider a simple comparative static when c falls.

First, an improved information technology leads to a declined of the aggregate job finding probability. As the cost of directing search falls, more workers apply to the productive firms and less workers apply to the unproductive firms. The reallocation of workers makes queues more unequal among firms. Due to the search frictions, the aggregate job finding probability falls. Interpreted from a matching efficiency perspective: The aggregate matching efficiency falls when cost of directing search falls.

Second, the average wage of the economy might rise or fall. There are two forces at work. The direct effect of falling cost is that firms are facing a more elastic application supply curve. To reach the same level of recruiting target, they have to promise a higher wage. This effect increases wage at all firms. The indirect effect comes from search friction. As workers are reallocated from unproductive to productive firms, the queueing in productive firms is even longer. The marginal worker is less valuable to the productive firms for recruiting purpose. This force drives down wage at the productive firms and drive up wages at the unproductive firms. The net effect depends on relative magnitude of the competition effect and congestion effect. This intuition is along the same line as Lester (2011). In Lester (2011), the congestion effect comes from firms shifting from accommodating informed searchers to uninformed searchers. In this model, it comes from reallocation of searchers from less congested to more congested markets.

5.2. *Simple Quantification*

The main takeaway of this paper is that the cost of directing search affects the rent sharing between market organizers and searchers and the efficient allocation of resources, in this paper firms and workers. For a quantitative study of this frictional trading environment, it is important to quantify the cost of directing search. The model in this paper does not have worker-side heterogeneity. So all the discussion in this section requires the researcher to either (1) select a set of homogeneous job searchers or (2) condition on observed worker heterogeneity.

Suppose the researcher can observe wages and the number of applicants at the same time, such as the studies using online job search boards. For example, assume the matching function is Cobb-Douglas, $n(q) = Aq^{\epsilon}$. By linearizing the labor supply curve in equation (8), an approximated elasticity of applicant-per-vacancy to the wage posted is:

$$\log q_j \approx \frac{1}{\frac{c}{m(q_j)w_j} + 1 - \epsilon} \log w_j. \tag{17}$$

Empirical studies on online job search behavior (e.g. Marinescu and Wolthoff, 2016) or experiments (e.g. Belot et al., 2018) find that the elasticity of applicant to wage is 0.7 to 0.9, meaning that a one percent increase in posted wage increases queue by $0.7 \sim 0.9$ percent. Although this paper assums away many potentially important mechanism in the job search process, equation (17) provides a way to interpret these elasticities. If we take the mid-range elasticity 0.8:

$$\frac{c}{m(q_j)w_j} = 1.25 - (1 - \epsilon).$$

This number is the the markdown due to cost of directing search as a fraction of wage. For example, if the $\epsilon = 0.5$, this result implies that in a large economy with homogeneous firms, the markdown due to cost of directing search would be 75% of the wage in equilibrium. It is important to note that this result does not rely on the static model. An extension to the dynamic model where workers only search when unemployed will result in the same link between the cost of directing search and queue-wage elasticity, with c scaled by the discount rate.

6. CONCLUSION

In this paper, I provide a tractable framework to study equilibrium implications of costly directed search. The cost of directing search is closely linked to the competition among firms and the efficiency of market equilibrium. Firms in an economy with a higher cost of directing search face a more inelastic labor supply curve. In the equilibrium, firms extract a markdown due to the cost of directing search. The markdown per se does not lead to inefficiency if it is equalized across firms with heterogeneous productivities. The market equilibrium is constrained efficient with a low cost of directing search. When the cost of directing search is high enough, workers' outside option prevents unproductive firms from lowering wages. As a result, the unproductive firms extract less markdown compared with the productive firms. Unequal markdowns across firms distort workers' search decisions, with the unproductive firms hire too often compared to the socially optimal allocation. I show the traditional remedy to the firm market power, the minimum wage, worsens the inefficiency by decreasing the markdown at unproductive firms even further. Instead, a self-balanced corporate profit transfer scheme can alleviate inefficiency while achieving the goal of redistribution from firms to workers.

To highlight the mechanism and its micro-foundation, I assumed away some realistic and salient features of frictional markets. Further exploring these possibilities would be interesting and crucial.

First, it would be important to incorporate worker heterogeneity in order to discuss the wage distribution and the efficiency of a market equilibrium. A two-sided heterogeneity model with costly directed search would introduce a new force that governs the sorting strength between workers and firms. Recent empirical studies found evidence of increasing sorting in developed countries. A possible explanation would be an improvement of information technology which allows workers to gather relevant information with a lower cost, and thus increases sorting.

Second, quantifying the cost of directing search should also be interesting. I discussed the methods to identify the cost of directing search given differfent types of datasets. To take the cost more seriously, introducing other types of frictions that might affect wages and allocations into the model is important.

Lastly, the compositional shift of job searchers and it interactions with firms' job creation incentives is an important hypothesis of the labor market fluctuations. The partially directed search model provides a framework that allows for flexible impacts compositional shift on job creation incentives. An interesting application would be to quantify the impacts of compositional shifts on labor market fluctuations using the partially directed search model.

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APPENDIX A: DETAILS OF $I \times J$ GAME

Definition 3 (Equilibrium in $I \times J$ economy)
A symmetric subgame perfect equilibrium is $\{\{q_j^e(\mathbf{w})\}_{j=1...J}, \mathbf{w^e}\}$*, such that:*
1. (Subgame Equilibrium)
$\forall\ \mathbf{w} = (w_1, ..., w_J)$

$$\{q_j^e(\mathbf{w})\}_{j=1...J} = \arg\max_{q_j} \quad \sum_{j=1}^{J} \frac{1-(1-q_j^e(\mathbf{w}))^I}{I q_j^e(\mathbf{w})} q_j \max\{w_j - b, 0\} - c \sum_{j=1}^{J} q_j \log \frac{q_j}{1/J}$$

s.t.

$$\sum_{j=1}^{J} q_j = 1$$

2. (Firm's Optimal Posting)

$$w_j^e = \arg\max_{w} \quad \Big(1-(1-q_j^e(\tilde{\mathbf{w}}))^I\Big)[z_j - w]$$

where

$$\tilde{w}_{j'} = \begin{cases} w_{j'}^e & if\ \ j' \neq j \\ w & if\ \ j' = j \end{cases}$$

The following proposition establishes the existence of a symmetric subgame perfect equilibrium and the equation system that characterize the equilibrium outcomes. Specifically, I look for a tuple of $\{w_j^e, q_j^e\}$ where (w_j, q_j) is the optimal choice of wage and queue for firm j, taking as given other firms' equilibrium wage postings and the labor supply curve.

Lemma 4 (Characterization of Symmetric Subgame Perfect Equilibrium)
$\{\mathbf{w}^e, \mathbf{q}^e\}$ *is the outcome of a symmetric subgame perfect equilibrium if and only if*
1. w_j^e *maximizes firm* j*'s profit given* $\mathbf{w}_{-j}^e$*:*

$$w_j^e = \arg\max_{w} \ [1-(1-q)^I](z_j - w)$$

s.t.

$$q = Q(w, \mathbf{w}_{-j}),$$

where $Q(w, \mathbf{w}_{-j})$ *is the solution to*

$$\frac{1-(1-q)^I}{Iq}(w-b)^+ - c\log\frac{q}{1/J} = V$$

for $j' \neq j$

$$\frac{1-(1-q_{j'})^I}{Iq_{j'}}(w^e_{j'}-b)^+ - c\log\frac{q_{j'}}{1/J} = V$$

$$q + \sum_{j'\neq j} q_{j'} = 1.$$

Solving for the equilibrium involves a non-trivial fixed-point problem. The complexity comes from strategic interactions between firms. If I investigate the equivalent problem in Lemma 3, the system involves simultaneously solving optimal wages because all firms internalize their impact on each other. Firms all understand their announcement will change workers' probability of applying to every firm. It is reasonable to conjecture that this kind of interaction would vanish if the number of firms grow to infinity, as the impact of each firm on other firms becomes small. This conjecture provides a second motivation to study a limiting economy where the population for both workers and firms grow to infinity. I will take on this task in Section 3.

Proposition 2.2. establishes the existence of such symmetric equilibrium. Here i sketch the intuition behind the existence of symmetric equilibrium. Workers' problem is almost identical to the 2×2 case, and a unique subgame equilibrium always exists. I can reduce the two-stage game into a normal form game where firms are given a labor supply curve that depends on the entire wage profile from other firms. The existence of a symmetric equilibrium hinges on firms adopting pure strategy in the equilibrium of this normal form game. The proof of proposition 2.2. shows that individual firm's payoff function is strictly concave in queue given other firms use pure strategy. I can thus use the standard results from normal form games to establishe the existence of a symmetric equilibrium.

APPENDIX B: MICRO-FOUNDATION: RATIONAL INATTENTION

This paper is motivated by the limited information in job search process. In this section, I make the link between a partially directed search model and limited information explicit. To do so, I first introduce an environment where workers face uncertainty about the wage posted by firms. Workers can reduce this uncertainty by acquiring information. Acquiring information is costly, where the cost is proportional to the reduction in uncertainty measured by Shannon's entropy. This type of learning model belong to the models of rational inattention, a booming literature since Sims (2003). I show a symmetric perfect Bayesian equilibrium in a posting game with uncertainty and with rational inattention can be solved as a collection of the symmetric subgame perfect equilibrium without uncertainty and with costly directed search. For simplicity of notation, I normalize $b = 0$.

B.1. *Equivalence between Information Acquisition and Costly Directed Search*

Setup – The production environment is identical to the case with the baseline model. There are I workers and J firms. Workers are indexed by $i = 1, ..., I$ and firms are indexed by $j = 1, ..., J$. Each firm has one vacant job to fill. When filled, the job at firm j produces output z_j. All agents have linear utility. If firm j hires a worker with wage w, the firm will receive a payoff of $z_j - w$ and worker will receive a payoff of w. For workers that fail to find a match, they receive their outside option of b. For firms that fail to find a match, they receive their outside option of 0.

The model with rational inattention differs in its information environment. Trades unfold in five stages. At the first stage, the vector of productivity of firms is $\mathbf{z} = (z_1, ..., z_J)$, drawn from i.i.d. distribution $G_Z(z)$, with finite support. At the second stage, firms observe the vector of productivities and decide on the wages when they hire a worker, given all the competitors' wages.[16] At the third stage, workers do not directly observe the wages offered by firms nor their productivities. They can learn

[16] It is an strong assumption to assume firms know the strategy of every competitor in the market; This assumption is less stark n in large economy, because firms only need to know the distribution of productivity in the limiting economy.

about the wages by paying a cost. Learning is modelled as a distribution of signals conditional on the actual wages. After observing the signals, workers make the decision of which firm to apply to maximize their expected payoffs. It is important to stress the information structure. Firms observe their own productivity and take as given other firms' wage postings. Workers cannot observe the productivity or the wage offered by firms. Both firms and workers understand the game: They understand the distribution of productivity, the optimization problem each agent is solving and their information availability. After the search decision is made, the matching stage and the hiring stage unfold as in the baseline cases. Workers can observe the wage when they are making offer acceptance decision.

Cost of Acquiring Information – Workers form a belief about the wages offered by firms $G(\mathbf{w}) \in \Delta W$, where ΔW is the set of Borel probability measure on $W = [0, \bar{w}]$[17]. The learning decision is a conditional distribution $F(\mathbf{s}|\mathbf{w})$ of signal $\mathbf{s}$. Signals are generated from the same space of wages: $\mathbf{s} \in \mathbb{R}^J$. I do not put any restriction on the conditional distribution $F(\mathbf{s}|\mathbf{w})$, other than it is a proper CDF ($\int_{\mathbf{s}} F(d\mathbf{s}|\mathbf{w}) = 1$). This conditional distribution models the information acquisition in job search. Workers might rely on various sources (e.g., LinkedIn and friends) to gather information about the compensation at different firms, and these sources provide some description of the wages. It might be noisy or precise. By putting effort into job search, workers can gather more precise information about the wages at different firms.

Reducing the uncertainty about wages requires efforts. The cost of acquiring information is proportional to the expected mutual information between the prior distribution of wages $G(\mathbf{w})$ and the posterior distribution of wages $F(\mathbf{w}|\mathbf{s})$. Mutual information is the difference between uncertainty evaluated at the two distributions, using Shannon's entropy:

$$\text{Cost of Acquiring Information} = c\left(\mathbb{H}(G) - E_{\mathbf{s}}\mathbb{H}(F_{\mathbf{w}|\mathbf{s}})\right),$$

$$\mathbb{H}(F) = -\int_{\mathbf{w}} f(\mathbf{w}) \log f(\mathbf{w}) d\mathbf{w}.$$

[17] I impose that wages are in bounded interval to make sure the expectations are defined.

Shannon's entropy is the expectation of negative logarithm of probability density, or the expected information of a distribution. Using the negative logarithm of probability to measure information satisfies four axioms of information (with a discrete distribution): (i) Monotonicity - more likely events contains less information, (ii) Non-negativity, (iii) Events with certainty do not provide information, and (iv) Additivity - Information from independent events are additive. Notice the mutual information is always weakly larger than 0 and is minimized when prior and posterior coincide.[18]

Equilibrium Definition – The equilibrium concept needs to be adapted to accommodate the uncertainty of workers regarding firms' wage postings. The natural concept is the *perfect Bayesian equilibrium* (hereafter, PBE):

Definition 4 (Symmetric Perfect Bayesian Equilibrium with Information Acquisition)

A ***Symmetric Perfect Bayesian Equilibrium*** *is a tuple* $\left\{G^e(\mathbf{w}), F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}^e_j(\mathbf{s})\}_j, w^e(z|\mathbf{z})\right\}$*:*

1. (Optimal Posting) $w^e(z|\mathbf{z})$ *maximizes the firm's profit given* $F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}^e_j(\mathbf{s})\}_j$ *and other firms use the same strategy, if the firm has productivity* z *and the entire productivity profile is realized at* $\mathbf{z}$*;*

2. (Optimal Search) $F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}^e_j(\mathbf{s})\}_j$ *maximizes every worker's payoff given belief* $G^e(\mathbf{w})$ *and other workers using the same strategy;*

3. (Consistency) $G^e(\mathbf{w})$ *is satisfies the Bayes rule given the productivity distribution* $G(\mathbf{z})$ *and* $w^e(z|\mathbf{z})$ *on the equilibrium path.*

In a PBE, workers form a belief regarding firms' equilibrium wage profiles according to Bayes rule. Workers optimally choose how to learn about the wage profile and how to apply for jobs based on their believes of the equilibrium wage profile. A subgame equilibrium is defined as a collection of learning and search strategy that maximize every worker's payoff given other workers' learning and search strategy, as well as the belief. Firms take as given the outcomes of subgames and determine their optimal wage postings. I focus on a symmetric perfect Bayesian equilibrium equilibrium where (i) workers adopt identical learning and search strategy and (ii) firms adopt pure strategy (firms with the same productivity post the same wage).

[18] A result from Jensen's inequality.

Subgame Equilibrium – First characterize the subgame given any belief of wages $G^e(\mathbf{w})$.[19] A symmetric subgame equilibrium is $\left\{F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}_j^e(\mathbf{s})\}_j\right\}$ such that every worker finds it optimal to adopt the equilibrium strategy when other workers do the same. Mathematically, it requires $\left\{F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}_j^e(\mathbf{s})\}_j\right\}$ solves the following fixed-point problem:

$$\left\{F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}_j^e(\mathbf{s})\}_j\right\} = \underset{F(\mathbf{s}|\mathbf{w}),\{\tilde{q}_j(\mathbf{s})\}_j}{\arg\max} \int_{\mathbf{w}} \int_{\mathbf{s}} \sum_j \frac{1-(1-q_j^e)^I}{I q_j^e} w_j \tilde{q}_j(\mathbf{s}) F(d\mathbf{s}|\mathbf{w}) G(d\mathbf{w}) - c\left(\mathbb{H}(G) - E_{\mathbf{s}}\mathbb{H}(F_{\mathbf{w}|\mathbf{s}})\right),$$

s.t.

$$\sum_j \tilde{q}_j(\mathbf{s}) = 1,$$

$$\int_{\mathbf{s}} F(d\mathbf{s}|\mathbf{w}) = 1$$

$$q_j^e(\mathbf{w}) = \int_s \tilde{q}_j^e(\mathbf{s}) F^e(d\mathbf{s}|\mathbf{w}).$$

Given other workers' use the equilibrium learning and search strategy $\{F^e(\mathbf{s}|\mathbf{w}), \tilde{q}_j^e(\mathbf{s})\}$, the probability of any individual worker applying to firm j is $q_j^e(\mathbf{w}) = \int_s q_j(\mathbf{s}) F(d\mathbf{s}|\mathbf{w})$. This probability is crucial for the link between a perfect Bayesian equilibrium with rational inattention and the subgame perfect equilibrium with observed wage postings and costly directed search. First, it summarizes the probability that any other worker applying to the same firm and thus is sufficient for the calculation of the job finding probability at every firm[20]. Second, it has the interpretation of a recommendation signal.

Suppose I restrict the feasible set of signal structures to be signals that directly suggest whether to apply to firm j. Workers are free to choose the probability of these recommendation signals, with the restriction that the probabilities of recommendations add up to 1. Restricting attention to these recommendation signal is without

[19] All workers face the same trading environment, so they form the same consistent belief about wage profile on the equilibrium path.

[20] Implicitly, I assume that workers do not receive correlated signals, so the law of large number holds.

loss of generality. In other words, any signal structure combined with optimal search decision can be represented as a recommendation signal directly telling workers where to apply that is always followed by workers. This result is similar to the logic of the revelation principle. In the setting of rational inattention, this result comes from two features of the learning and search decision: (1) The conditional distribution of signals and the search decision enter multiplicatively into the expected income and (2) the cost of acquiring information from Shannon's entropy has a chain rule.[21] This result has intuitive economic interpretation: workers do not gather new information in the search stage. The recommendation strategy carries the same amount of information content as the signal structure that induces workers to behave as the recommendation strategy.

Matějka and McKay (2015) states this result in a decision theory context. They show that a decision problem with uncertainty of payoffs and entropy cost in acquiring information can be solved as a decision problem with observed payoffs and a cost based on the mutual information between the recommendation signal and a baseline probability. The baseline probability is the search probability according to the prior. Lemma 5 restated their results in a frictional environment.

Lemma 5 (Matějka and McKay (2015) with Search Friction)
$\{F(\mathbf{s}|\mathbf{w}), \{\tilde{q}_j(\mathbf{s})\}_j\}$ *solve worker's problem given the equilibrium recommendation strategy* $q_j^e(\mathbf{w})$ *if and only if the recommendation strategy* $q_j(\mathbf{w}) = \int_{\mathbf{s}} \tilde{q}_j(\mathbf{s}) F(d\mathbf{s}|\mathbf{w}) d\mathbf{s}$ *solves the following problem*

$$\max_{\{q_j(\mathbf{w})\}} \int_{\mathbf{w}} \sum_j \frac{1-(1-q_j^e(\mathbf{w}))^I}{I q_j^e(\mathbf{w})} w_j q_j(\mathbf{w}) G(d\mathbf{w}) - c\left(\mathbb{H}(\bar{q}) - \int_{\mathbf{w}} \mathbb{H}(q(\mathbf{w})) dG(\mathbf{w})\right),$$

s.t.

$$\sum_j q_j(\mathbf{w}) = 1,$$

$$\bar{q}_j = \int_{\mathbf{w}} q_j(\mathbf{w}) G(d\mathbf{w}).$$

[21] For two random variables X and Y, conditional information of X on Y equals the mutual information of (X, Y) minus the information of Y.

Two assumptions in this paper further simplify this problem: (1) Firms draw productivities from the same distribution; (2) In a symmetric equilibrium, firms with the same productivity adopt an identical strategy. As a result, a priori firm identity does not reveal any information regarding wages in a symmetric equilibrium. Uniform distribution $\frac{1}{J}$ measures the information content in prior distribution. In another word, in a symmetric equilibrium, the benchmark probability is $\bar{q}_j = \frac{1}{J}$. In the context of search theory, this is the random search strategy. Reduction in uncertainty is measured by the difference of entropy from random search strategy and the actual strategies after information acquisition.

Lemma 5 is a powerful result. I can essentially reduce the strategy space of worker's problem from a high-dimensional space of distribution into the search probability among J firms, for a consistent belief of equilibrium wage distribution $G(\mathbf{w})$. Additionally, in a symmetric equilibrium with identical productivity distribution, I can analyze the subgames node-by-node, because the benchmark strategy $\bar{q}_j$ does not depend on the belief $G(\mathbf{w})$.

Corollary 7.1 (Subgame Recommendation Equilibrium)
$\{F(\mathbf{s}|\mathbf{w}), \{q_j(\mathbf{s})\}_j\}$ *is a subgame equilibrium given* $G(\mathbf{w})$ *in a symmetric perfect Bayesian equilibrium if and only if it solves the following fixed point problem*

$$q_j^e(\mathbf{w}) = \arg\max_{\{q_j(\mathbf{w})\}} \int_{\mathbf{w}} \sum_j \frac{1-(1-q_j^e(\mathbf{w}))^I}{I q_j^e(\mathbf{w})} w_j q_j(\mathbf{w}) G(d\mathbf{w}) - c \int_{\mathbf{w}} \sum_j q_j(\mathbf{w}) \log \frac{q_j(\mathbf{w})}{1/J} dG(\mathbf{w}),$$

s.t.

$$\sum_j q_j(\mathbf{w}) = 1$$

Proof: Take the results of 5; Write out the mutual information using $q_j(\mathbf{w})$ and benchmark $\frac{1}{J}$. *Q.E.D.*

The subgame recommendation equilibrium already looks very similar to the subgame equilibrium in an economy where workers observe the wages and pay a cost to direct search. One caveat: there is no restriction on the wage profiles that are with zero probability given $G(\mathbf{w})$. For wage profiles that are off-equilibrium, the definition of a perfect Bayesian equilibrium does not put any restriction on worker's belief. Because workers do not think these wage profiles are possible, they will never gather information about these non-existent wage profiles. Worker's search decision might be ill-informed if firms actually deviate to those wage profiles. The search decisions at those ill-informed states might prevent firms from actually deviating to those states. One could construct multiple equilibria using this logic. The same issue arises in Bayesian games without information acquisition. A remedy is to introduce the trembling-hand refinement and require the subgame equilibrium restrictions to hold off-equilibrium. Ravid (2019) provide a definition of such refinement in the context of information acquisition.

Symmetric Perfect Recommendation Equilibrium – I adopt the equilibrium refinement as in Ravid (2019). Formally, it requires that for any wage profile, I can find some perturbation that visits this wage profile with positive probability and the subgame recommendation equilibrium is still an equilibrium given such a perturbation.

Definition 5 (Perfect Recommendation Equilibrium with Information Acquisition) *A symmetric* ***Perfect Recommendation Equilibrium*** *is a symmetric* ***Perfect Bayesian Equilibrium*** *such that*

1. (credible response) $q^e_j(\mathbf{w})$ *is a credible response to* $\{G^e(\mathbf{w}), w^e(z|\mathbf{z})\}$*:*

For every $\mathbf{w} = (w_1, ..., w_J)$*, there exists a sequence* $(G^n, \{\sigma^n_j\}_j)_n$ *such that:*

a. $\sigma^n_j(w_j|\mathbf{z}) > 0$ *for every* $\mathbf{z}$ *and every* $j-th$ *element of* $\mathbf{w}$*;*

b. $\sigma^n_j(w|\mathbf{z})$ *converges strongly to* $\delta_{w(z_j|\mathbf{z})}$ *for every* $\mathbf{z}$

c. G^n *is consistent with* $\{\sigma^n_j\}_j$*;*

d. For all n*,* $q^e_j(\mathbf{w})$ *is the subgame recommendation equilibrium given* G^n*.*

2. (Attentive) There exits $\mathbf{w}$ *such that workers apply to every firm with positive probability.*

Proposition 9 is the key result of this section: An allocation is the subgame recommendation equilibrium in a symmetric perfect recommendation equilibrium if and only if for every wage (on or off the equilibrium path), this allocation is also the subgame equilibrium in a game with observed wage profiles and the cost of directing search.

Proposition 8
$q_j^e(\mathbf{w})$ is a credible response to $\{G^e(\mathbf{w}), w^e(z|\mathbf{z})\}$ if and only if for every fixed $\mathbf{w}$, $q_j^e(\mathbf{w})$ is the solution to the subgame equilibrium with costly directed search and observed wage $\mathbf{w}$.

PROOF: See Appendix. *Q.E.D.*

Firms face identical information environment in the game with observed wage profiles and the game with information acquisition. From proposition 9, I show the subgame equilibrium outcomes in the game with rational inattention and that in the game with observed wage profile and costly directed search are identical. Given the solution to the subgame equilibrium is unique, firms in the two games face identical problem. In conclusion, a symmetric perfect recommendation equilibrium can be solved as a collection of subgame perfect equilibrium with observed wage profiles and cost of directing search, given different productivity vectors.

Corollary 8.1 (Equivalence between symmetric perfect recommendation equilibrium and equilibrium with costly directed search)
$\{G^e(\mathbf{w}), q_j^e(\mathbf{w}), w^e(z|\mathbf{z})\}$ is a symmetric recommendation equilibrium if an only if for every $\mathbf{z}$, $(\{q_j^e(\mathbf{w})\}_j, w(z|\mathbf{z}))$ is a symmetric subgame perfect equilibrium with full information and costly directed search.

Corollary 8.2 (Existence of Symmetric perfect recommendation equilibrium)
A symmetric perfect recommendation equilibrium exits.

PROOF: Proposition 2 establishes that a symmetric subgame perfect equilibrium with costly directed search exists for any productivity $\mathbf{z}$. A symmetric subgame perfect equilibrium with full information and costly directed search is also a symmetric perfect recommendation equilibrium by corollary 9.1. *Q.E.D.*

APPENDIX C: ALTERNATIVE COST FUNCTIONS

So far, I focus on the K-L divergence as the cost of directing search, because of its foundation in information theory and its tractability. To show this framework is generalizable to other parametrization of cost functions, I now analyze the partially directed search for a general class of divergence measures called f-divergence. Specifically, the f-divergence takes the Radon-Nikodym derivative between the chosen search strategy and the uniform distribution a_j and evaluate the integral of the following form:

$$\text{Cost of Directing Search} = \int_0^1 \phi(a_j)dj$$

where ϕ is increasing and convex. To ensure non-deviation is costless, $\phi(1) = 0$. The f-divergence nests the K-L divergence as a special case when $\phi(a) = a \log a$. Now, I show partially directed search can also be motivated by the general f-divergence, the entropic competitive search equilibrium can be similarly defined, and the inefficiency of market equilibrium also exists. For simplicity, I assume $b = 0$ and $\mu = 1$.

Partially Directed Search with f-divergence Consider the worker's problem given any wage function ω and queue function $q(\omega)$. Workers' problem is as in equation (18). The worker's problem is again convex: it has a strictly concave objective function and linear constraint. The optimal search decision implies the search strategy must be a solution to condition (11). The marginal cost of applying to firm j is now measured by $c\phi'(q_j)$.

$$\max_a \int_0^1 m(q_j(\omega))\omega_j^+ q_j dj - c\int_0^1 \phi(q_j)dj \tag{18}$$

s.t.

$$\int_0^1 q_j dj = 1$$

$$m(q_j(\omega))\omega_j^+ - c\phi'(q_j) = V \tag{19}$$

Entropic Competitive Search Equilibrium with f-divergence The equilibrium with the general cost function can be accordingly defined: I look for wages that maximize firm's profit, search strategy that maximizes workers' payoff, and the equilibrium supply curve that is consistent with worker's decisions. I directly state the equivalent problem in equation Definition 6.

Definition 6 (Equivalent problem with f-divergence)

1. firm's optimality given V

$$\{q(V), w(V)\} = \max_{w,q} n(q)(z_j - w)$$

s.t.

$$m(q)w - c\phi'(q) = V$$

$$w \geq 0$$

2. Market Clearing

$$\int_0^1 q(V^e)dj = 1$$

Proving the existence of uniqueness follows the same logic as the case with K-L divergence: Firm's problem is strictly convex given any V. The optimal choice of queue is decreasing in V with at least a positive measure of firms that is strictly decreasing. Therefore, law of demand holds for the aggregate demand of applicants. However, the result on wage is different for the general class of cost function. Specifically, in equation (13), the markdown due to the cost of directing search depends on the curvature of cost function, which is zero when $\phi(a) = a \log a$.

$$\max\{n'(q_j^e)z_j - c\phi''(q_j^e), 0\} - c\phi(q_j^e) = V^e$$

$$w_j = \max\{\frac{n'(q_j)}{m(q_j)}z_j - c\frac{\phi''(q_j)}{m(q_j)}, 0\} \tag{20}$$

Efficiency with f-divergence The planner's problem is defined in equation (14). The constrained efficient allocation equalizes the benefit and cost of applying to firm j. Comparing this allocation the allocation from the market equilibrium, I find the markdown due to cost of directing search has impact on efficiency. The curvature in cost of directing search is crucial for how monopsony power distorts allocation to firms. With K-L divergence, all firms extract a constant markdown in expectation. In the general case, the markdown differs for firms with heterogeneous productivities. However, the inefficiency due to incentive compatibility constraint of workers prevails in the general case: In the equilibrium with general f-divergence, there can be cases where positive measure of firms post workers' outside option. Given this result, our discussion about minimum wage holds for general cost functions.

$$\max_{q_j} \int_0^1 n(q_j)z_j dj - c\int_0^1 \phi(q_j)dj \tag{21}$$

s.t.

$$\int_0^1 q_j dj = 1$$

$$q_j = q_j$$

$$n'(q_j^*) - c\phi'(q_j^*) = V^*$$

APPENDIX D: PARTIALLY DIRECTED SEARCH AND STANDARD MONOPSONY MODELS

This section rewrites the firms' problem in terms of expected hiring n instead of queue length. To do so serves two purpose. First, it makes clear the origins of market power in a partially directed search model. Second, this steps relate a partially directed search model to a standard model of monopsony. The partially directed search model provides an alternative interpretation of the labor supply elasticity estimated from these models. To prepare the notations, I define the the inverse function of the job-filling probability to be $g(n)$. This function maps the expected hirings into the number of application per firm:

$$g(N) = n^{-1}(N).$$

Let $W(n; V, c)$ be the wage a firm needs to pay if it plans to hire n workers when the market utility is V. Inverting the optimal decision of workers I get a labor supply curve:

$$W(n; V, c) = b + \frac{g(n)}{n}\bigg(V + c \log g(n)\bigg). \tag{22}$$

Given the market utility, firm j's problem is as in equation 22. Firm j decides on the number of workers to hire. Hiring n workers produces nz_j and incurs a total labor cost of $nW(n; V, c)$. A constraint is in place for the firm's problem. It reflects the fact workers always have the option to walk away from potential matches. So firms can never hire any one if they promise a wage below b. This formulation makes it clear the role of outside option in the partially directed search model: Workers' outside option is a potentially binding "minimum" wage.

$$\max_{n} \quad z_j n - nW(n; V, c),$$

s.t.

$$W(n; V, c) \geq b.$$

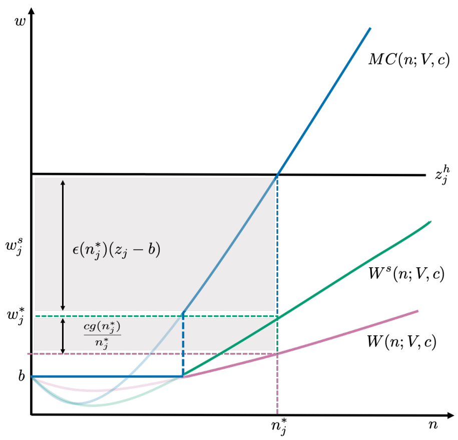
To analyze the firm's problem, I first derive the marginal cost curve $MC(n; V, c)$:

$$MC(n; V, c) = W(n; V, c) + nW'(n; V, c) = b + g'(n)\Big(V + c\log g(n)\Big) + g'(n)c$$

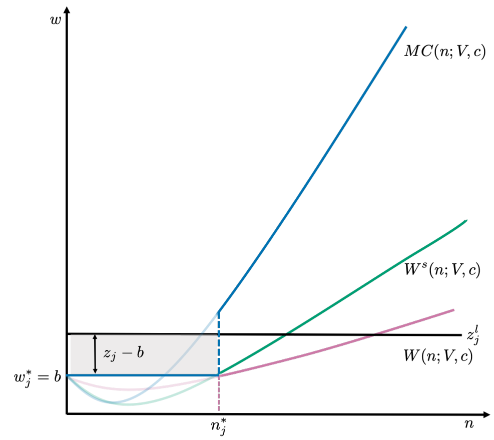
The marginal cost has two components, the first component reflects the contribution of a marginal worker to the matching process at firm j. In order to hire extra worker, the firm has to compensate workers their outside option b and their contribution to the matching process. Due to the cost in directing search, the red component reflects the market power due to cost of directing search. I will define $W^s(n; V, c)$ as the social value component:

$$W^s(n; V, c) = W(n; V, c) + nW'(n; V, c) = b + g'(n)\Big(V + c\log g(n)\Big).$$

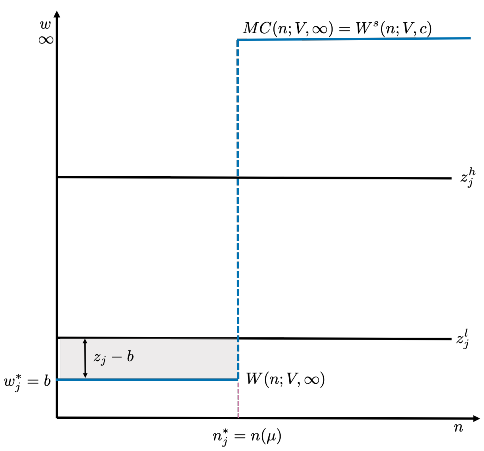
Figure 5 plots the labor supply and labor demand curve given fixed V. Panel (a) is the case with a positive cost of directing search for a productive firm. The labor supply curve is in pink. Any wage below worker's outside option results in zero employment, so the labor supply curve is bounded below by b. For the employment that results in a wage above b, the labor supply curve is upward-sloping. (The segment below is not necessarily increasing due to the functional form of K-L divergence; however, it does not matter because of the constraint). The social value function of marginal worker W^s is in green and is above the labor supply curve, reflecting the search friction in the economy. To hire additional worker, the firm has to attract more than one applicants, which changes the matching probability of everyone applying to the same firm. The marginal cost of hiring additional worker, MC, is further above W^s reflecting the cost of directing search. There is a discontinuous jump of marginal cost curve. For the units of employment lower than threshold $AC = b$, all units must be hired at a constant



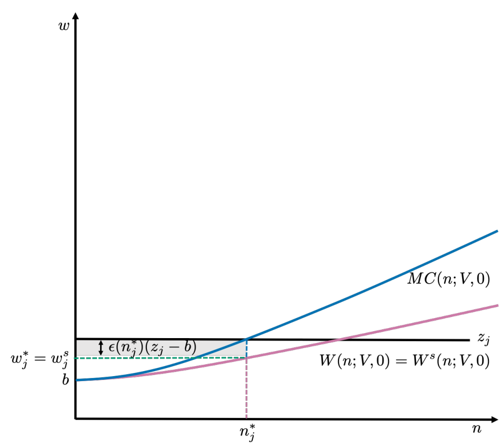
(a) Productive Firms with $c > 0$



(b) Unparoductive Firms with $c > 0$



(c) $c = \infty$



(d) $c = 0$

Figure 3: Supply Demand Diagram for Fixed V

cost (b), so there is no distortion between marginal and average cost. Beyond this threshold, the marginal cost jumps up because in order to hire more the firm offers a higher wage, for every job searcher in the market.

To find the optimal input of the firm, I look for the crossing point of marginal productivity z_j and the marginal cost of recruiting MC. The level of employment is optimal for firm j because the marginal benefit equals the marginal cost. The wage offered to workers is the wage required to hire n_j^* workers. The wage is below marginal product for two reasons: (1) the markdown due to search friction. Firms need to be compensated for creating jobs (the gap between MC curve and W^s curve); (2) the markdown due to the cost of directing search (the gap between W^s and W curve). It is evident that the markdown due to the oligopoly competition vanishes in the entropic competitive search equilibrium.

Panel (b) plots the input decision of an unproductive firm for the same cost of directing search. For this firm, productivity is low enough that the optimal employment is at the threshold $AC = b$. At this point, social value and wage function coincide, because there is the first unit that firm j decides to post higher wage in order to hire more. Firm j takes all the gains from trade and workers get paid their outside option.

Panel (c) plots the case $c \to \infty$. In this case, distorting search is infinitely costly. To hire more workers than the exogenous queue μ, the firm needs to post an infinite wage, which can never be optimal if $y_j < \infty$. In this case, both firms (productive and unproductive) offer workers' outside option, and extract all the gains from trade.

Panel (c) plots the case $c \to 0$. In this case, the labor supply curve and the social value curve always coincide. Firms and workers both get their contributions to the matching process. There is no bunching at workers' outside option because a relatively more productive firm can always differentiate itself from an unproductive firm. Because search is costless, posting a slightly higher wage attracts strictly more workers.

APPENDIX E: PROOF OF THEOREMS

Proof of Lemma 1

Want to show the following equation system has a unique solution:

$$(1 - q_1 + \frac{q_1}{2})(w_1 - b)^+ - c \log q_1 = (1 - q_2 + \frac{q_2}{2})(w_2 - b)^+ - c \log q_2,$$

$$q_1 + q_2 = 1.$$

Using the second equation to write the system solely in q_1:

$$\frac{2 - q_1}{2}(w_1 - b)^+ - \frac{1 + q_1}{2}(w_2 - b)^+ - c \log \frac{q_1}{1 - q_1} = 0.$$

The left hand side is continuous and strictly decreasing in q_1.
When $q_1 = 0$, the LHS is

$$(w_1 - b)^+ - \frac{1}{2}(w_2 - b)^+ + \infty > 0$$

When $q_1 = 1$, the LHS is

$$\frac{1}{2}(w_1 - b)^+ - (w_2 - b)^+ - \infty < 0$$

So there is one and only one solution to this equation in the interval $[0, 1]$. There is a unique $(q_1, 1 - q_1)$ that solves the original equation. Interchange the role of firm 1 and firm 2 results in $(1 - q_1, q_1)$ as the solution. So the subgame equilibrium is independent of firm identities conditional on wages.

Proof of Proposition 1

The goal is to look for the solution to the following equation system in terms of

$$(q_1, q_2, w_1, w_2, \gamma_1, \gamma_2)$$

(Subgame Equilibrium)

$$c \log \frac{q_1}{q_2} = \left(\frac{2 - q_1}{2}\right)(w_1 - b) - \left(\frac{2 - q_2}{2}\right)(w_2 - b)$$

(Firm 1's Optimality)

$$(1-q_1)(z_1-b)+\gamma_1=\left(\frac{1}{2}+q_1\right)(w_2-b)+c\left(1+\frac{q_1}{1-q_1}+\log\frac{q_1}{1-q_1}\right)$$

(Firm 2's Optimality)

$$(1-q_2)(z_2-b)+\gamma_2=\left(\frac{1}{2}+q_2\right)(w_1-b)+c\left(1+\frac{q_2}{1-q_2}+\log\frac{q_2}{1-q_2}\right)$$

(Probability)

$$q_1+q_2=1$$

(participation constraint at Firm 1)

$$\gamma_1\geq 0\perp w_1-b\geq 0$$

(participation constraint at Firm 2)

$$\gamma_2\geq 0\perp w_2-b\geq 0$$

Consider four cases, depending on whether the participation constraints are binding.

Case 1: $\gamma_1,\gamma_2=0$

In this case, I look for solution to the following equation:

$$T(q_1)=0$$

where

$$\begin{aligned}T(q)=&\frac{2-q}{3-2q}\left(q(z_2-b)-c\left(1+\frac{1-q}{q}+\log\frac{1-q}{q}\right)\right)\\&-\frac{1+q}{1+2q}\left((1-q)(z_1-b)-c\left(1+\frac{q}{1-q}+\log\frac{q}{1-q}\right)\right)\\&-c\log\frac{q}{1-q}\end{aligned}$$

$$T'(q)>0$$

$$\lim_{q\to 0}T(q)=-\infty$$

$$\lim_{q\to 1} T(q) = \infty$$

$T(q)$ crosses zero once and only once. Given this unique q, the rest of equilibrium objects (q_2, w_1, w_2) are uniquely pinned down. I can find a solution in this case if both firms are posting positive wages:

$$T_1(q_1) \geq 0$$

$$T_2(q_1) \geq 0$$

where

$$T_1(q) = (1-q)(z_1 - b) - c(1 + \frac{q}{1-q} + \log\frac{q}{1-q})$$

$$T_2(q) = q(z_2 - b) - c(1 + \frac{1-q}{q} + \log\frac{1-q}{q})$$

Case 2: $\gamma_1 > 0, \gamma_2 = 0$

In this case, $w_1 = b$, the solution is characterized by firm 2's optimal posting condition given firm 1 posts b:

$$0 = T_2(q) = q(z_2 - b) - c\left(1 + \frac{1-q}{q} + \log\frac{1-q}{q}\right)$$

There is a unique solution to this equation. Denote this solution $\tilde{q}$. I can find a solution in this case if firm 2 is posting a positive wage and the marginal value of increasing wage is negative for firm 1:

$$\tilde{q}(z_2 - b) - c\left(1 + \frac{1-\tilde{q}}{\tilde{q}}\right) \geq 0$$

$$\tilde{T}(q; c) < 0$$

where $\tilde{T}(q)$ is the marginal benefit for firm 1 to increase wage given firm 2 is posting optimally:

$$\tilde{T}(\tilde{q}; c) = (1-q)(z_1 - b) - c\left(1 + \frac{q}{1-q} + \log\frac{q}{1-q}\right) - c\frac{1+2q}{1+q}\log\frac{1-q}{q}$$

Case 3: $\gamma_1 = 0, \gamma_2 > 0$

In this case, $w_2 = b$,

$$0 = T_1(q)$$

Similar to case 2 I can solve for a unique q, denoted as $\tilde{q}$. I can find a solution in this case if:

$$(1-\tilde{q})(z_1-b) - c\left(1+\frac{\tilde{q}}{1-\tilde{q}}\right) \geq 0$$

$$q(z_2-b) - c(1+\frac{1-\tilde{q}}{\tilde{q}} + \log\frac{1-\tilde{q}}{\tilde{q}}) - c\frac{3-2\tilde{q}}{2-\tilde{q}}\log\frac{\tilde{q}}{1-\tilde{q}} < 0$$

Case 4: $\gamma_1 > 0, \gamma_2 > 0$

In this case,

$$q_1 = q_2 = \frac{1}{2}$$

$$w_1 = w_2 = b$$

This is the equilibrium outcome if

$$c \geq \frac{\max\{z_1, z_2\} - b}{4}$$

Existence and uniqueness: In every case, I can find a unique solution. To prove existence, I check that at least one of the four cases hold given any (z_1, z_2, c, b). To prove uniqueness, I check there is only one case that holds. WLOG, assume $z_2 > z_1$. I will divide the parameters according to the cost of directing search. First define two thresholds of cost: $\bar{c}_1$ and $\bar{c}_2$.

High-Threshold - $\bar{c}_2$: This is the threshold where the productive firm stops to post wage above the outside option b. For the costs larger than this threshold, I can only find equilibrium in case 4 where both firms post $w_j = b$.

$$\bar{c}_2 = \frac{z_2 - b}{4}$$

Low-Threshold - $\bar{c}_1$: This is the threshold where the unproductive firm stops posting wage above b. To look for this threshold, I solve the following equation in c:

$$\tilde{T}(\tilde{q}(\bar{c}_1); \bar{c}_1) = 0$$

where $\tilde{q}(\bar{c}_1)$ solves

$$T_2(\tilde{q}; \bar{c}_1) = 0$$

Economically, $\bar{c}_1$ is the cost such that when firm 2 posts optimally given firm 1 posts $w_1 = b$, the optimal strategy for firm 1 is posting wage $w_1 = b$. I can always find a unique solution $\bar{c}_1$ because of the following two statements:

Statement 1: $\tilde{T}(\tilde{q}(c); c)$ is strictly decreasing in c. To show this, I total differentiate $\tilde{T}(\tilde{q}(c); c)$ w.r.t c:

$$\begin{aligned}
\frac{d\tilde{T}(\tilde{q}(c); c)}{dc} &= \frac{\partial \tilde{T}}{\partial q}\frac{dq}{dc} + \frac{\partial \tilde{T}}{\partial c} \\
&= -\frac{1}{1-q} - \frac{q}{1+q}\log\frac{1-q}{q} \\
&- \frac{q(1+q\log\frac{1-q}{q})(2cq(1+q) + c(1-q)^2\log\frac{1-q}{q} + (1-q^2)^2(z_1-b))}{(1-q)(1+q)^2(c+(1-q)q^2(z_2-b))} \\
&< 0
\end{aligned}$$

The last inequality comes from evaluating the expression at $q < \frac{1}{2}$ (This is the relevant range when $c < \frac{z_2-b}{4}$).

Statement 2:

$$\tilde{T}(\tilde{q}(0); 0) = z_1 - b > 0$$

$$\tilde{T}(\tilde{q}(\frac{z_2-b}{4}); \frac{z_2-b}{4}) = \frac{1}{2}(z_1-b) - 2c \leq \frac{1}{2}(z_2-b) - 2c = 0$$

So I can find a unique $\bar{c}_1$ that solves the equation. When $c < \bar{c}_1$, $\tilde{T}(\tilde{q}(\bar{c}_1); \bar{c}_1) > 0$ and when $c > \bar{c}_1$, $\tilde{T}(\tilde{q}(\bar{c}_1); \bar{c}_1) < 0$

Next, I investigate the cost of directing search by thresholds.

a. When $c > \bar{c}_2$ I already showed an equilibrium of case 4 exists. WTS: the other cases cannot be an equilibrium. Notice when $c > \bar{c}_2$:

$$T_1(\frac{1}{2}) < 0$$

$$T_2(\frac{1}{2}) < 0$$

Case 1: To find an equilibrium of case 1, it must be $\tilde{q} < \frac{1}{2}$. By the monotonicity of T_2, I have $T_2(\tilde{q}) < T_2(\frac{1}{2}) < 0$. A contradiction to case 1's conditions.

Case 2: To find an equilibrium of case 2, it must be $\tilde{q} > \frac{1}{2}$. The subgame equilibrium condition implies that $b = w_1 > w_2$. A contradiction to case 2's condition

$w_2 > w_1 = b$. This cannot be an equilibrium.

Case 3: To find an equilibrium of case 3, it must be $\tilde{q} < \frac{1}{2}$. The subgame equilibrium condition implies that $b = w_2 > w_1$. A contradiction to case 3's condition $w_1 > w_2 = b$. This cannot be an equilibrium.

b. When $c \in [\bar{c}_1, \bar{c}_2)$, he following inequality holds:

$$\tilde{T}(\tilde{q}(c), c) \leq \tilde{T}(\tilde{q}(\bar{c}_1), \bar{c}_1) = 0$$

At $\tilde{q}(c)$, firm 2 is posting positive wage and the marginal benefit of increasing wage is negative for firm 1. Thus, I have found an equilibrium in of case 2.

Case 1: I can always $T(\hat{q}; c) = 0$, this point has to be $\hat{q} < q(c)$, at this point $T_2(\hat{q}; c) < 0$.

Case 3: To find an equilibrium I solve $T_1(q) = 0$. When $c > \bar{c}_1$, the solution has to be $\tilde{q} > \tilde{q}(c)$, at which point firm 2 is willing to post a wage above b.

Case 4: is directly ruled out by the condition.

c. When $c < \bar{c}_1$, want to show the only equilibrium must be in case 1. Notice given this cost of directing search c:

$$\tilde{T}(\tilde{q}(c), c) > \tilde{T}(\tilde{q}(\bar{c}_1); \bar{c}_1) = 0$$

So

$$(1 - q(c))(z_1 - b) - c(1 + \frac{q(c)}{1 - q(c)} + \log \frac{q(c)}{1 - q(c)}) - c\frac{1 + 2q(c)}{1 + q(c)} \log \frac{1 - q(c)}{q(c)} > 0$$

Combined with $T(q)$'s definition, at $q(c)$

$$T(q; c) < 0$$

So I can find $\hat{q} > q(c)$ such that $T(\hat{q}; c) = 0$. Next, verify given $\hat{q}$, both firms are willing to post wages above outside option of workers.

$$T_2(\hat{q}; c) > T_2(q(c); c) = 0$$

So in this case, firm 1 must be posting wage above outside option. Given $z_2 > z_1$, the solution to $T(q) = 0$ must be below $\frac{1}{2}$. This means firm 2 is posting a wage $w_2 > w_1 > b$. This is indeed an equilibrium of case 1.

Case 2 cannot be equilibrium, because given c, the equilibrium will be $\tilde{q}(c)$, at which point firm 1 is willing to post wage above outside option.

Case 3 cannot be equilibrium, because in order to find $T_1(\tilde{q}) = 0$, $\tilde{q} < q(c) < \frac{1}{2}$. This means firm 2 is posting a wage above firm 1, a contradiction.

In conclusion:

- $c \geq \bar{c}_2$: the unique equilibrium is in which both firms post b;
- $c \in [\bar{c}_1, \bar{c}_2)$: the unique equilibrium is in which the productive firm post $w > b$ and the unproductive firm post b;
- $c < \bar{c}_1$: the unique equilibrium is in which both firms post $w > b$;

Proof of Corollary 1.1

Guess $w_1^e = w_2^e = w^*$, for firm j the profit maximization problem is

$$\max_w (1 - (1-q)^2)(z - w)$$

s.t.

$$(1 - q + \frac{q}{2})(w - b) - (q + \frac{1-q}{2})(w^* - b) = c\frac{q}{1-q}$$

Writing the problem in terms of q instead w, I have

$$\max_q (1 - (1-q)^2)(z - b) - 2q(c \log \frac{q}{1-q} + (q + \frac{1-q}{2})(w^* - b))$$

s.t.

$$w \geq b$$

Taking the first order condition w.r.t q

$$(1-q)(z-b) - (c \log \frac{q}{1-q} + (q + \frac{1-q}{2})(w^* - b)) - q(c\frac{1}{q} + c\frac{1}{1-q} + \frac{1}{2}(w^* - b)) + \gamma = 0$$

where γ is the multiplier on $w \geq b$.
Imposing $q = \frac{1}{2}$ and $w = w^*$ I get

$$\frac{1}{2}(z^* - b) - \frac{3}{4}(w^* - b) - \frac{1}{2}(4c + \frac{1}{2}(w^* - b)) + \gamma = 0$$

Rearrange we get

$$w^* = b + \max\{\frac{z-b}{2} - 2c, 0\}$$

The equation we used to derive w^* is the definition of a symmetric equilibrium: each firm optimize given the other firm's strategy; firms with identical productivity use the same strategy. As w^* is the only solution, the symmetric equilibrium allocations and wages are unique.

Proof of Corollary 1.2

More productive firm posts higher wage and attracts longer queue; **Higher cost leads to less queue differential.**
First consider the case when both firm 1 and firm 2 post wage above b: Following the proof of proposition 1, the equilibrium probability of applying to firm 1 solves

$$T(q_1) = 0$$

For the first statement of Corollary 1.2, it suffices to check $T(\frac{1}{2})$. Using the formula from proposition 1, I reach

$$T(\frac{1}{2}) = \frac{3}{8}(z_2 - z_1)$$

If $z_1 > z_2$, $T(\frac{1}{2}) < 0$. Because I showed $T'(q) > 0$ and there is only an unique equilibrium, this means $q_1^* > \frac{1}{2} > q_2^*$. From worker's optimal search condition, it must be such that $w_1^* > w_2^*$.

For the second statement of proposition 3, it suffices to check $\frac{\partial T}{\partial c}$. Differentiate $T(q)$, I get:

$$\frac{\partial T}{\partial c} = \frac{2\left(-2q^3 + 3q^2 + q + (1-q)q\log\left(\frac{q}{1-q}\right) - 1\right)}{(1-q)q(3-2q)(2q+1)}$$

Notice this expression is independent of (z_1, z_2). The sign of the derivative takes

$If\ q > \frac{1}{2}$

$$\frac{\partial T}{\partial c} > 0$$

$If\ q < \frac{1}{2}$

$$\frac{\partial T}{\partial c} < 0$$

$If\ q = \frac{1}{2}$

$$\frac{\partial T}{\partial c} = 0$$

Take the case $z_1 > z_2$ as example, I already established that $q_1 > \frac{1}{2}$ in this case. Now suppose for c_2, $T(q_1(c_2)) = 0$. At this point, $T_c > 0$. So $T(q_1(c_2); c_1) > 0$. Thus it cannot be an equilibrium. Using the monotonicity of $T(q)$, it must be some $q_1(c_1) < q_1(c_2)$ that satisfies the equilibrium conditions.

Now consider the case when only one firm post wage $w > b$: In this case, the firm that post wage above b solves the problem

$$\max_w (1-(1-q)^2)(z_j - w)$$

s.t.

$$(1 - q + \frac{q}{2})(w - b) = c \log \frac{q}{1-q}$$

Writing in terms of q I get

$$\max_q (1-(1-q)^2)(z_j - b) - 2q(c \log \frac{q}{1-q})$$

The F.O.C. is

$$(1-q)(z-b) - c \log \frac{q}{1-q} - c(1 + \frac{q}{1-q}) = 0$$

If $w > b$,

$$\log \frac{q}{1-q} > 0$$

This implies the more productive firm (the one that posts wage above b) attracts longer queue than the unproductive firm (the one that posts wage b). The F.O.C. also implies an increase in c leads to a decrease in q.
The case where both firms post b automatically satisfies the statements.
The proof of proposition 1 already showed there are two thresholds.

E.1. *Proof of Proposition 2*

To show the symmetric equilibrium exits, I want to show the payoff of a firm j choosing queue length q is quasi-concave in its argument. More specifically, firm j solves the following problem given other firms' wage postings: (To simplify the notation, normalize $b = 0$. The case for $z_j > b > 0$ can be accordingly derived.)

$$\max_{w \geq 0} (1 - (1-q)^I)(z_j - w)$$

s.t.

$$\frac{1-(1-q)^I}{Iq} w - c(\log \frac{q}{1/J} + 1) = V$$

$$\frac{1-(1-q_{j'})^I}{Iq_{j'}} w_{j'} - c(\log \frac{q_{j'}}{1/J} + 1) = V$$

$$\sum_{j'} k_{j'} q_{j'} - q_j + q = 1$$

We can write firm's problem in terms of probability q by eliminating w:

$$\max_{q} \Pi(q) = (1-(1-q)^I) z_j - Iq(V + c \log Jq)$$

s.t.

$$\frac{1-(1-q_{j'})^I}{Iq_{j'}} w_{j'} - c \log \frac{q_{j'}}{1/J} = V$$

$$\sum_{j'} k_{j'} q_{j'} - q_j + q = 1$$

$$V + c \log Jq \geq 0$$

First investigate the shape of the profit function:

$$\Pi'(q) = I(1-q)^{I-1}z - IV - cI\log Jq - cI - Iq\frac{dV}{dq}$$

$$\Pi''(q) = -I(I-1)(1-q)^{I-2}z - c\frac{I}{q} - I\frac{dV}{dq} - Iq\frac{d^2V}{dq^2}$$

Other terms are straightforward, except for the response of market utility V to q. Differentiate the equation system that defines q and V I have:

$$\frac{dV}{dq} = -\frac{1}{\sum_{j'\neq j}(\xi_{j'})^{-1}} > 0$$

where

$$\xi_j = X_j w_j - \frac{c}{q_j} < 0$$

X_j is the response of job finding probability at firm j to change in other worker's strategy:

$$X_j = \frac{I^2 q_j(1-q_j)^{I-1} - I(1-(1-q_j)^I)}{I^2 q_j^2} < 0$$

Further differentiate V w.r.t. q we have:

$$\frac{d^2V}{dq^2} = -\frac{\sum_{j'\neq j}(\xi_{j'})^{-2}\frac{d\xi_{j'}}{dq}}{\left(\sum_{j'\neq j}(\xi_{j'})^{-1}\right)^2}$$

where

$$\frac{d\xi_{j'}}{dq} = (\frac{dX_j}{dq}w_j + \frac{c}{q_j^2})\frac{dq_{j'}}{dq}$$

The sign of $\frac{d^2V}{dq^2}$ depends on whether $\frac{d\xi_{j'}}{dq}$ is positive. Because the job finding probability is convex in q:

$$\frac{dX_j}{dq} > 0$$

An increase in q leads to decreases in $q_{j'}$:

$$\frac{dq_{j'}}{dq} < 0$$

Thus we have $\frac{d\xi_{j'}}{dq} < 0$ and $\frac{d^2V}{dq^2} > 0$. This result has a very natural economic interpretation: it is marginally more costly to attract workers when the queue is already long, because the search friction is more severe and the cost of directing search is convex. Because $\frac{dV}{dq} > 0$ and $\frac{dV^2}{dq^2} > 0$, we have:

$$\Pi''(q) = -I(I-1)(1-q)^{I-2}z - c\frac{I}{q} - I\frac{dV}{dq} - Iq\frac{d^2V}{dq^2} < 0$$

Combining all the results I have shown that the payoff function of any firm is strictly concave given other firms use pure strategy. However, there is a constraint on the feasible q. I now show the feasible set is convex. Suppose q^1 and q^2 are in the feasible set. For $\alpha \in [0,1]$ and q^1 and q^2, WTS:

$$\alpha q^1 + (1-\alpha)q^2 \geq \min\{q^1, q^2\}$$

WLOG assume $q_2 = \min\{q^1, q^2\}$. Because both V and $\log q$ are increasing in q:

$$V|_{\alpha q^1 + (1-\alpha)q^2} + c\log J(\alpha q^1 + (1-\alpha)q^2) \geq V|_{q_2} + c\log Jq_2 \geq 0$$

Thus the feasible set is convex. I have shown that the payoff function is strictly concave on a convex set. Thus the solution is unique. Utilizing standard results from Debreu (1952), a pure strategy equilibrium exist in the first stage game. Given any wage profile, a unique symmetric subgame perfect equilibrium exists. Thus, there exists at least one symmetric subgame prefect equilibrium.

E.2. *Proof of Corollary 2.1*

Suppose all firms use identical strategy w^*. For each individual firm j, the problem is

$$\max_q (1-(1-q)^I)(z-w)$$

s.t.

$$\frac{(1-(1-q)^I)}{Iq}(w-b) - c\log Jq = \frac{(1-(1-\frac{1-q}{J-1})^I)}{I\frac{1-q}{J-1}}(w^*-b)^+ - c\log J\frac{1-q}{J-1}$$

Rewrite the problem in terms of q:

$$\max_q (1-(1-q)^I)(z-b) - Iq(c\log Jq + \frac{(1-(1-\frac{1-q}{J-1})^I)}{I\frac{1-q}{J-1}}(w^*-b)^+ - c\log J\frac{1-q}{J-1})$$

Take first-order condition and impose $q = \frac{1}{J}$ I get

$$\left(\frac{1-(1-\frac{1}{J})^I}{I/J} - \frac{1}{J}\frac{1}{J-1}\frac{I(1-\frac{1}{J})^{I-1}}{I\frac{1}{J}} + \frac{1}{J}\frac{1}{J-1}\frac{1-(1-\frac{I}{J})^I}{I/J^2}\right)(w^*-b)$$
$$= (1-\frac{1}{J})^{I-1}(z-b) - c - \frac{c}{J-1}$$

Imposing $\frac{I}{J} = \mu$:

$$\left(\frac{J}{J-1}\frac{1-(1-\frac{I}{J})^{\mu J}}{\mu} - \frac{1}{(J-1)}(1-\frac{1}{J})^{\frac{I-1}{I}\mu}\right)(w^*-b)$$
$$= (1-\frac{1}{J})^{\frac{I-1}{I}\mu J}(z-b) - c - \frac{c}{J-1}$$

Rearrange and impose the constraint $w \geq b$ I reach

$$w^* = b + \max\{\mu\frac{(1-\frac{1}{J})^{\mu J}(z-b)-c}{1-(1-\frac{1}{J})^{\mu J} - \frac{1}{J-1}(1-\frac{1}{J})^{\mu J}\mu}, 0\}$$

E.3. *Proof of Proposition 3*

The equivalent problem is the same as looking for a Walrasian equilibrium. I look for V^* such that the optimal demand for applicants equal to the exogenous supply of workers. To show the existence, I rely on the continuity of demand function. To show uniqueness, I rely on the strict monotonicity of demand function. Focus on the firm's problem given any V

$$\max_{w\geq 0, q} n(q)[z_j - w]$$

s.t.

$$m(q)(w-b)^+ - c[\log q - \log\mu] = V$$

The demand for applicant is solution to the following equation

$$\max\{n'(q)z_j - c, 0\} - c[\log q - \log\mu] = V$$

For the case $n'(q)z_j > c$, differentiate the condition yields

$$\frac{dq}{dV} = \frac{1}{n''(q)z_j - c\frac{1}{q}} < 0$$

For the case $n'(q)z_j < c$, differentiate the condition yields

$$\frac{dq}{dV} = \frac{1}{-c\frac{1}{q}} < 0$$

For the case $n'(q)z_j = c$, the equation above is not differentiable. However, both left and right derivative points to a decrease in demand if V increase

$$\frac{dq}{dV}|_{+} = \frac{1}{n''(q)z_j - c\frac{1}{q}} < 0$$

$$\frac{dq}{dV}|_{-} = \frac{1}{-c\frac{1}{q}} < 0$$

Thus the law of demand holds for all firms. The solution to the first order condition is continuous in V by Maximum Theorem. I next show I can always find a unique solution for $V^* \in [0, \max_j z_j]$.

When $V = 0$, the constraint implies

$$q_j^*(0) \geq \mu$$

When $V = \max_j z_j$, the maximization implies there is no firm posting wage above its productivity. This means $w_j^* \leq z_j \leq \max_j z_j$. From the constraint

$$c[\log q_j - \log \mu] = m(q)w_j^+ - \max_j z_j \leq w_j^+ - \max_j z_j \leq 0$$

The first inequality uses $m(q) \leq 1$. So I reach

$$q_j^*(\max_j z_j) \leq \mu$$

The aggregate demand for applicants $D(V) = \int_0^1 q_j^*(V)dj$ is strictly decreasing in V because the integrand is strictly decreasing in V. It has the property

$$D(0) \geq \mu$$

$$D(\max_j z_j) \leq \mu$$

So there exists a unique $V^* \in [0, \max_j z_j]$ such that $D(V^*) = \mu$

E.4. *Proof of corollary 3.2*

For every z. Solve the following equation

$$n'(\bar{q})(z-b) = c \tag{23}$$

Denote the solution to this equation as $\bar{q}(c,z)$. $\bar{q}$ is decreasing in c because n is concave. For every z, I find c such that the market clears when z is exactly constrained by the $w \geq b$. For $z_j > z$

$$-c \log \bar{q}(c,z) = n'(q_j)(z_j - b) - c \log q_j$$

For $z_j < z$

$$q_j = \bar{q}(c,z)$$

All q_j are strictly decreasing and continuous in c. Using the same logic of finding an entropic competitive search equilibrium. If $n(q)$ has the Inada condition, there is a unique $c \in (0,\infty)$ such that

$$\int_0^1 q_j dj = \mu$$

Call this cost $\bar{c}(z)$. Next I want to show that $\bar{c}$ is increasing in z. First, for every level of cost c, $\bar{q}$ is increasing in z

$$\frac{\partial \bar{q}}{\partial z} = -\frac{n'(\bar{q})}{n''(\bar{q})(z-b)} > 0$$

Suppose there are two productivity level $z_1 > z_2$, if for z_2, $\bar{c}_1$ is the threshold that clears the market. Then $\bar{c}_1$ cannot be the threshold for z_1, because for c_1, $\bar{q}$ increased and the aggregate queue is lower than μ. To clear the market, $\bar{q}$ must decrease. For this to happen, $\bar{c}_2 > \bar{c}_1$.

E.5. *Proof of Lemma 3*

The proof of lemma is an application of Maximum Theorem. I already show that individual firm's problem can be written as an optimization problem with convex feasible set and strictly concave objective function. To make notation explicit, let's write the problem in terms of t, the population size.

$$\Pi^*(t, \mathbf{w}^t, \mathbf{q}^t) = \max_q (1-(1-q)^{tI})(z_j - b) - tIq(V + c\log tJq)$$

s.t.

$$\frac{1-(1-q_{j'})^{tI}}{tIq_{j'}}(w_{j'} - b) - c\log\frac{q_{j'}}{1/(tJ)} = V_j$$

$$t\sum_{j'} k_{j'}q_{j'} - q_j + q = 1$$

$$V + c\log tJq \geq 0$$

I established earlier that the objective function is strictly concave when $c > 0$ and the feasible set is convex. According to Maximum Theorem, the optimal solution $q_j(t, \mathbf{w}^t, \mathbf{q}^t)$ is single-valued and continuous function in $(t, \mathbf{w}^t, \mathbf{q}^t)$ and V, and value at optimum $\Pi^*(t, \mathbf{w}^t, \mathbf{q}^t)$ is also continuous in $(t, \mathbf{w}^t, \mathbf{q}^t)$. Now I just need to take limit of the stationary conditions to derive the limit. Given any vector of wage $(t, \mathbf{w}^t, \mathbf{q}^t)$, the best response of a firm of productivity z_j is $(q^*, \{q_{j'}\}_{j'}, V^*)$ that solves the following equations:

$$\Pi'(q^*) = (1-q^*)^{tI-1}z_j - V^* - c\log tJq - c - q^*\frac{dV}{dq}|_{q^*} = 0$$

For all j':

$$\frac{1-(1-q_{j'})^{tI}}{tIq_{j'}}(w_{j'} - b) - c\log\frac{q_{j'}}{1/(tJ)} = V^*$$

and lastly

$$t\sum_{j'} q_{j'} - q_j + q = 1$$

To make the notation explicit for the limit, define $Q = tIq$. Using $\mu = \frac{I}{J}$, the $J + 2$ equation system for optimal posting decision becomes:

$$(1 - \frac{1}{tI}Q^*)^{tI-1}z_j - V^* - c\log\frac{Q^*}{\mu} - c - \frac{1}{tI}Q^*\frac{dV}{dq}|_{q^*} = 0$$

For all j':

$$\frac{1-(1-\frac{1}{tI}Q_{j'})^{tI}}{Q_{j'}}(w_{j'}-b)-c\log\frac{Q_{j'}}{\mu}=V^*$$

and lastly

$$\frac{1}{\mu}\sum_{j'}Q_{j'}-\frac{1}{tI}Q_j+\frac{1}{tI}Q^*=1$$

The key observation for the convergence result is that the first order condition is only linked to the constraints by V^*, the market utility. The first step is to show, as $t\to\infty$, the impact of every firm on the market utility vanishes:

$$\frac{1}{tI}Q^*\frac{dV}{dq}|_{q^*}\to 0$$

Q^* is always a bounded number (it is a probability) and $\frac{1}{tI}$ asymptotes to 0. For now I ignore the case with binding constraint. The proof of proposition 2 establishes that

$$\frac{dV}{dq}=-\frac{1}{\sum_{j'\neq j}(\xi_{j'})^{-1}}$$

where

$$\xi_j=X_j(w_j-b)-\frac{c}{q_j}<0$$

X_j is the response of job finding probability at firm j to change in other worker's strategy,

$$X_j=\frac{(tI)^2q_j(1-q_j)^{tI-1}-tI(1-(1-t_j)^{tI})}{(tI)^2q_j^2}<0$$

First let's rewrite X_j as

$$X_j=\frac{q_j(1-q_j)^{tI-1}-\frac{1}{tI}(1-(1-q_j)^{tI})}{q_j^2}$$

Using the harmonic mean inequality:

$$\frac{dV}{dq}=-\frac{1}{\sum_{j'\neq j}(\xi_{j'})^{-1}}\leq\frac{1}{tJ-1}\frac{1}{tJ-1}\sum_{j'\neq j}(-\xi_{j'})$$

I have

$$-\frac{1}{tJ-1}\xi_j = \frac{c}{(tJ-1)q_j} - \frac{(1-q_j)^{tI-1}}{(tJ-1)q_j} + \frac{1}{(tJ-1)q_j}\frac{(1-(1-q_j)^{tI})}{tIq_j}$$
$$\leq \frac{c}{(tJ-1)q_j} + \frac{1}{(tJ-1)q_j}\frac{(1-(1-q_j)^{tI})}{tIq_j}$$
$$\leq \frac{c+1}{(tJ-1)q_j}$$
$$\leq \frac{c+1}{(tJ-1)\underline{Q}}$$

The first inequality comes from $-\frac{(1-q_j)^{tI-1}}{(tJ-1)q_j} < 0$; The second inequality comes from $\frac{(1-(1-q_j)^{tI})}{tIq_j} \leq 1$; The third inequality comes from q_j shares a common lower bound $\underline{Q}$, where $\underline{Q}$ is the probability of searching other firms if firm j posts z_j and other firms post b. To find this probability:

$$-c\log\underline{Q} = \frac{1-((tJ-1)\underline{Q})^{tI}}{I(1-(tJ-1)\underline{Q})}(z_j-b) - c\log(1-(tJ-1)\underline{Q}))$$

Because $\frac{1-((tJ-1)\underline{Q})^{tI}}{I(1-(tJ-1)\underline{Q})}$ is a probability and is bounded by 1:

$$c\log(1-(tJ-1)\underline{Q})) - c\log\underline{Q} = \frac{1-((tJ-1)\underline{Q})^{tI}}{I(1-(tJ-1)\underline{Q})}(z_j-b) \leq (z_j-b)$$

Rearranging the LHS:

$$c\log(\frac{1}{\underline{Q}} - (tJ-1)) \leq (z_j-b)$$

Taking exponential on both sides:

$$\frac{1}{\underline{Q}} - (tJ-1) \leq e^{\frac{z_j-b}{c}}$$

Rearranging:

$$(tJ-1)\underline{Q} \geq \frac{tJ-1}{tJ-1+e^{(z_j-b)/c}}$$

Thus

$$\frac{dV}{dq} \leq \frac{c+1}{(tJ-1)\underline{Q}} \leq \frac{1+c}{1+\frac{1}{tJ-1}e^{(z_j-b)/c}}$$

As $t \to \infty$:

$$\lim_{t\to\infty} \frac{dV}{dq} \leq 1 + c$$

Therefore:

$$\lim_{t\to\infty} \frac{1}{tI}\frac{dV}{dq} = 0$$

The first order condition in the limit becomes

$$(1 - e^{-Q^*})z_j - V^* - c \log Q^* - c = 0$$

where V is the limit of sequence of V^t that solves the constraints on firm j's problem. There are another $J+1$ equations for $Q_{j'}$ and V^*, as $t \to \infty$ they become:

$$\frac{1 - e^{-Q_{j'}}}{Q_{j'}}(w_j - b) - c \log \frac{Q_{j'}}{\mu} = V^*$$

$$\sum_j Q_j = \mu$$

The assumption for this lemma implies $(tI\mathbf{q}^t, V_\infty)$ satisfy the equation system above. We can find a unique Q^* that solves:

$$(1 - e^{-Q^*})z_j - V_\infty - c \log Q^* - c = 0$$

So $(Q^*, tI\mathbf{q}^t, V_\infty)$ solve the limit version of the first order conditions. This is also the unique solution to the equation system because given $\mathbf{w}^t$, there is a unique solution to the system (it is a subgame with wage vector $\mathbf{w}^t$).

$$\frac{1 - e^{-Q_{j'}}}{Q_{j'}}(w_j - b) - c \log \frac{Q_{j'}}{\mu} = V^*$$

$$\sum_j Q_j = \mu$$

Using the continuity of the solution to firm's problem, it has to be such that the limit of firm's optimal choice converges to the solution to the equation above. The above condition is also the unique solution to the firm's problem in the limit taking as given V_∞:

$$\max_w (1 - e^{-Q})(z_j - w)$$

s.t.

$$\frac{1-e^{-Q}}{Q}(w-b) - c\log\frac{Q}{\mu} = V_\infty$$

E.6. *Proof of Proposition 4*

1. **There exists at least one convergent sub-sequence of subgame perfect equilibrium**

To prove this point, consider any sequence of subgame perfect equilibrium $\{\mathbf{w}_t^e, \mathbf{q}_t^e\}_t$, the marginal benefit of search for every firm must be equal in the equilibrium:

$$V_t = \frac{1-(1-q_{t,j}^e)^{tI}}{Iq_{t,j}^e}(w_{t,j}^e - b) - c\log\frac{q_{t,j}^e}{tJ}$$

So the t-economies are indexed by V_t, in that given any V_t and t, I can solve for the equilibrium allocation and wage. V_t is a real number that is bounded, because the marginal benefit of applying to firms cannot be positive infinity (due to the cost of directing) or negative infinity (due to the wage is bounded by b). The Bolzano–Weierstrass theorem implies there must exist a convergent subseuqence of $\{V_t\}_t$. Find the corresponding allocations and wages to this subsequence. Lemma 3 states that the allocations and wages must converges to the optimal queue-wage choice of firms in an entropic competitive search equilibrium, given market utility $V_\infty = \lim_{t\to\infty} V_t$.

Now I show V_∞ must clears the market, so the corresponding queues and wages are indeed an entropic competitive search equilibrium. The proof relies on the continuity of firms' problem in both finite and limiting economy. There is always an equilibrium with the same distribution of productivities and worker-firm ratio μ. Define the market utility of this equilibrium V^e. Suppose that $V_\infty \neq V^e$. Denote $q_j(V)$ the solution to firms of productivity z_j‘ problem in the limiting economy given market utility V. The entropic competitive search equilibrium is unique. If $V_\infty \neq V^*$, then $t\sum_j q_j(V_\infty) \neq \mu$. Using continuity of q_j^t in t, for every t, I can find some $t' > t$ such that

$$\sum_j q_j^t(V_{t'}) \neq 1$$

This is contradicting to the fact $\{\mathbf{w}^e_{t'}, \mathbf{q}^e_{t'}\}$ is a subgame perfect equilibrium. So the limiting point must be $V_\infty = V^e$. So the limiting of this subsequence is indeed an entropic competitive search equilibrium. Moreover, because the limiting equilibrium is unique, every convergent sequence limits to the same point.

2. **Every entropic competitive search equilibrium is a limit of some convergent sequence**

Our proof for statement 1 is constructive. So following the same method I can find the convergent sequence that limits to any entropic competitive search equilibrium with discrete productivity distribution.

E.7. *Proof of Proposition 5*

Firm's problem is strictly convex, and so is continuous in V and y. Given any fixed V, denote firm's choice of queue given z as $q(z;V)$. Convergence in productivity distribution implies $\int_z q(z;V)dF^n(z) \to \int_z q(z;V)dF^*(z)$.
Define $D^n(V) = \int_z q(z;V)dF^n(z)$ and $D^*(V) = \int_z q(z;V)dF^n(z)$. $D_n(V)$ and $D^*(V)$ are continuous bijections from $[0, \max_j z_j]$ to $[0, \mu]$. Their inverse $D_n^{-1}(\mu)$ is also continuous. Let $\{V_n\}_n$ be the market utility with distribution F_n and V^* is the market utility with distribution F^*. It must be such that $V_n \to V^*$. Otherwise I can find n large enough such that V_n does not clears the market.
Define a new random variable $Z_n = (z_n, V_n)$. $Z_n \xrightarrow{d} (z^*, V^*)$. Continuous mapping theorem establish that

$$q_n = q(z_n, V_n) \xrightarrow{d} q^*$$

$$w_n = w(z_n, V_n) \xrightarrow{d} w^*$$

E.8. *Proof of Proposition 7*

I first consider the impact of minimum wage on the market utility V. It has to be increasing in the minimum wage. To see this, notice the firms who are constrained by the minimum wage has a queue that solves the following equation

$$m(\bar{q})\underline{w} - c\log\bar{q} = V$$

If the market utility weakly decreases when the minimum wage increases. $\bar{q}$ will increase. For a weakly decreasing market utility, the queues at unconstrained firms weakly increase. As a result, the aggregate demand for applicants strictly increases. Given the old market utility clears the market and the equilibrium is unique, this is a contradiction.

For the unconstrained firms, this first order condition is

$$n'(q)z - c\log q - c = V$$

I already showed the market utility must rise when the minimum wage increases. The first order condition implies that the queue for the unconstrained firms must decrease.

Next, I show the threshold of productivity must increases. Suppose to the contrary, the threshold decreases. This means some firms that used to be bounded by the minimum wage is now unconstrained, including the old threshold firms. Recall the threshold firms find minimum wage optimal. Our discussion so far implies, the queue at the old threshold firm must be decreasing. The new threshold is less productive than the old threshold firm, and thus post a lower queue than old minimum wage queue. Therefore, weakly less firms are posting the minimum wage, and the queue they demand from market is decreasing. This cannot be true in equilibrium, because now the aggregate demand for applicants decreases for all firms.

Next I turn to the results on wages: Posted wages increase for all firms. For the firms that use to post the old minimum, their wages increase mechanically. For the unconstrained firms, their wages increase. To see this, notice the wage for an unconstrained firm with productivity z_j is

$$w_j = \max\{\epsilon(q_j)z_j - \frac{c}{m(q_j)}, 0\}$$

I have shown q_j decreases for these firms. The matching elasticity $\epsilon(q_j)$ is decreasing, and $m(q_j)$ is decreasing in q_j. A decrease in q_j leads an increase in wage.

The last is to show the wage at the newly constrained firms also increase. To see this, notice these firms use to be unconstrained and are now posting minimum wage. Suppose these firms are posting a wage higher than before. The new threshold firm used to be unconstrained and more productive than the newly constrained firms. The new threshold firm must used to post a wage higher than these firms, and thus higher than the new minimum wage. This is a contradiction, because I just showed the threshold firm must be increasing its wage.

E.9. *Proof of Proposition 8*

First assume the tax function is well-behaved, such that it maintains the strict concavity of the objective function. Take the first order condition given tax function $T(\pi)$ I have

$$n'(q)z_j - (V^e + c\log q) - c - n'(q)T(\pi)) + (c - \frac{qm'(q)}{m(q)}(V^e + c\log q))T'(\pi) = 0$$

Comparing this equation to planner's solution, I notice the wedge is

$$-c - n'(q)T(\pi)) + (c - \frac{qm'(q)}{m(q)}(V + c\log q))T'(\pi)$$

The goal is to set tax policy function such that for every q, given the equilibrium market utility replicates planner's solution V^*, the wedge is zero. According to the labor supply curve $V + c\log q = m(q)w$. The tax function needs to be such that

$$c + n'(q)T(\pi) = (c - qm'(q)w)T'(\pi)$$

With this wedge being zero, the wage must be

$$m(q)w = n'(q)z$$

This implies

$$\pi = z - w = -\frac{qm'(q)}{n'(q)}w$$

So the tax function must be such that

$$\frac{c}{n'(q)} + T(\pi) = (\frac{c}{n'(q)} + \pi)T'(\pi)$$

To make sure this exactly cancelled the wedge at every point. I impose that equilibrium market utility is $V^e = V^*$. For every q and its associated π. To show the tax function is convex, differentiate $T'(\pi)$:

$$\begin{aligned} T''(\pi) &= \frac{T'\pi - T + \frac{c}{n'}(T'-1) - c\frac{n''q'}{(n')^2}(\pi - T)}{(\pi + \frac{c}{n'(q(\pi))})^2} \\ &= \frac{\frac{-T\frac{c}{n'}}{\pi+\frac{c}{n'}} + \frac{c}{n'}\frac{T-\pi}{\pi+\frac{c}{n'}} - c\frac{n''q'}{(n')^2}(\pi - T)}{(\pi + \frac{c}{n'(q(\pi))})^2} \\ &= \frac{\frac{\frac{c}{n'}\pi-T\frac{c}{n'}}{\pi+\frac{c}{n'}} + \frac{c}{n'}\frac{T-\pi}{\pi+\frac{c}{n'}} - c\frac{n''q'}{(n')^2}(\pi - T)}{(\pi + \frac{c}{n'(q(\pi))})^2} \\ &= \frac{-c\frac{n''q'}{(n')^2}(\pi - T)}{(\pi + \frac{c}{n'(q(\pi))})^2} > 0 \end{aligned}$$

Proof of Proposition 9

I want to show that a symmetric perfect Bayesian equilibrium is indeed a Perfect Recommendation Equilibrium. To do so, I just need to check the two refinements are met.

Costly Directed Search to Rational Inattention

Want to show if $q^e(\mathbf{w})$ solves the following mapping for every $\mathbf{w}$, then it must be credible and attentive.

$$\{q^e_j\}_j = \arg\max_{q_j} \sum_j \frac{1-(1-q^e_j)^I}{Iq^e_j} w_j q_j - c\sum q_j \log\frac{q_j}{1/J} \tag{24}$$

s.t.

$$\sum_j q_j = 1$$

1. Credible Response

Consider a fixed $\mathbf{w}$, the goal is to construct a sequence of believes and firm strategy G^n, σ^n_j such that $\mathbf{w}$ is visited with positive probability. With these believes, the subgame equilibrium in a symmetric perfect Bayesian equilibrium is still subgame equilibrium with the new belief. Our strategy is to construct G_n such that the firms stay identical ex'ante. To do so: For any perturbation wage profile $\mathbf{w} \geq 0$, let $\delta_{\mathbf{w}}$ be the delta function with parameter $\mathbf{w}$. For any equilibrium wage profile $\mathbf{w}^e$, define the perturbation firm strategy:

$$\sigma_j^n = \left(1 - \frac{1}{n}\right)\delta_{w^e} + \frac{1}{n}\left(\sum_{k=1}^{J} \frac{1}{J}\delta_{\tilde{w}^k}\right)$$

where $\tilde{\mathbf{w}}^k$ is constructed by a shuffling of perturbation wage profile $\mathbf{w}$. Specifically, I can construct such perturbation by shuffling firm identities in the following matrix. Each column has the same element, but I sequentially shuffle to leading elements to the tail. Call the k-th column of following matrix the wage profiles $\tilde{w}^k$.

$$\begin{bmatrix} w_1 & w_2 & w_3 & \dots & w_J \\ w_2 & w_3 & w_4 & \dots & w_1 \\ \dots & \dots & \dots & \dots & \dots \\ w_J & w_1 & w_2 & \dots & w_{J-1} \end{bmatrix}$$

The potential issue with an arbitrary wage perturbation to $\mathbf{w}$ is that the perturbation to $\mathbf{w}$ might induce the benchmark strategy (the strategy before acquiring information) to be non-uniform. In such a case, using uniform search strategy as the benchmark strategy no longer characterize the optimal strategy. My solution is to find a collection counter-perturbations such that the benchmark strategy stays uniform. I construct such perturbation by shuffling firm identities. Given the wage profiles, the corresponding subgame equilibrium in the game with observed wages and cost of directing search must be the following matrix

$$\begin{bmatrix} q_1^e & q_2^e & q_3^e & \dots & q_J^e \\ q_2^e & q_3^e & q_4^e & \dots & q_1^e \\ \dots & \dots & \dots & \dots & \dots \\ q_J^e & q_1^e & q_2^e & \dots & q_{J-1}^e \end{bmatrix}$$

No matter what is the perturbation profile $\mathbf{w}$, the row sum of the above matrix must be 1. So by uniformly weight the counter-perturbation profiles, I maintain a uniform benchmark strategy. This means equation (23) still characterize the subgame perfect equilibrium given the perturbated firm strategy and belief.

2. Attentive

For wage profile that equalize across all firms, it is always a subgame equilibrium to put positive weight on all firms. So the equilibrium outcome is attentive.

Rational Inattention to Costly Directed Search On the equilibrium path, a symmetric equilibrium means two firms use the same strategy if they have the same strategy. Because productivity distribution is identical across firms, this means the belief on wage profile must be identical across firms on the equilibrium path. Thus if any benchmark strategy is the equilibrium outcome, it must be uniform. For workers' strategy to be credible, corollary 8.1 implies the equation (23) must hold for every $\mathbf{w}$.

SUPPLEMENT: PARTIALLY DIRECTED SEARCH IN THE LABOR MARKET

1. MICRO-FOUNDATION: RATIONAL INATTENTION

This paper is motivated by the limited information in job search process. In this section, I make the link between a partially directed search model and limited information explicit. To do so, I first introduce an environment where workers face uncertainty about the wage posted by firms. Workers can reduce this uncertainty by acquiring information. Acquiring information is costly, where the cost is proportional to the reduction in uncertainty measured by Shannon's entropy. This type of learning model belong to the models of rational inattention, a booming literature since Sims (2003). I show a symmetric perfect Bayesian equilibrium in a posting game with uncertainty and with rational inattention can be solved as a collection of the symmetric subgame perfect equilibrium without uncertainty and with costly directed search. For simplicity of notation, I normalize $b = 0$.

1.1. *Equivalence between Information Acquisition and Costly Directed Search*

Setup – The production environment is identical to the case with the baseline model. There are I workers and J firms. Workers are indexed by $i = 1, ..., I$ and firms are indexed by $j = 1, ..., J$. Each firm has one vacant job to fill. When filled, the job at firm j produces output z_j. All agents have linear utility. If firm j hires a worker with wage w, the firm will receive a payoff of $z_j - w$ and worker will receive a payoff of w. For workers that fail to find a match, they receive their outside option of b. For firms that fail to find a match, they receive their outside option of 0.

The model with rational inattention differs in its information environment. Trades unfold in five stages. At the first stage, the vector of productivity of firms is $\mathbf{z} = (z_1, ..., z_J)$, drawn from i.i.d. distribution $G_Z(z)$, with finite support. At the second stage, firms observe the vector of productivities and decide on the wages when they hire a worker, given all the competitors' wages.[1] At the third stage, workers do not directly observe the wages offered by firms nor their productivities. They can learn about the wages by paying a cost. Learning is modelled as a distribution of signals conditional on the actual wages. After observing the signals, workers make the decision of which firm to apply to maximize their expected payoffs. It is important to stress the information structure. Firms observe their own productivity and take as given other firms' wage postings. Workers cannot observe the productivity or the wage offered by firms. Both firms and workers understand the game: They understand the distribution of productivity, the optimization problem each agent is solving and their information availability. After the search decision is made, the matching stage and the hiring stage unfold as in the baseline cases. Workers can observe the wage when they are making offer acceptance decision.

Cost of Acquiring Information – Workers form a belief about the wages offered by firms $G(\mathbf{w}) \in \Delta W$, where ΔW is the set of Borel probability measure on $W = [0, \bar{w}]$[2]. The learning decision is a conditional distribution $F(\mathbf{s}|\mathbf{w})$ of signal $\mathbf{s}$. Signals are generated from the same space of wages: $\mathbf{s} \in \mathbb{R}^J$. I do not put any restriction on the conditional distribution $F(\mathbf{s}|\mathbf{w})$, other than it is a proper CDF ($\int_{\mathbf{s}} F(d\mathbf{s}|\mathbf{w}) = 1$). This conditional distribution models the information acquisition in job search. Workers might rely on various sources (e.g., LinkedIn and friends) to gather information about the compensation at different firms, and these sources provide some description of the wages. It might be noisy or precise. By putting effort into job search, workers can gather more precise information about the wages at different firms.

Reducing the uncertainty about wages requires efforts. The cost of acquiring information is proportional to the expected mutual information between the prior distribution of wages $G(\mathbf{w})$ and the posterior distribution of wages $F(\mathbf{w}|\mathbf{s})$. Mutual information is the difference between uncertainty evaluated at the two distributions,

[1]It is an strong assumption to assume firms know the strategy of every competitor in the market; This assumption is less stark n in large economy, because firms only need to know the distribution of productivity in the limiting economy.

[2]I impose that wages are in bounded interval to make sure the expectations are defined.

using Shannon's entropy:

$$\text{Cost of Acquiring Information} = c\left(\mathbb{H}(G) - E_{\mathbf{s}}\mathbb{H}(F_{\mathbf{w}|\mathbf{s}})\right),$$

$$\mathbb{H}(F) = -\int_{\mathbf{w}} f(\mathbf{w}) \log f(\mathbf{w}) d\mathbf{w}.$$

Shannon's entropy is the expectation of negative logarithm of probability density, or the expected information of a distribution. Using the negative logarithm of probability to measure information satisfies four axioms of information (with a discrete distribution): (i) Monotonicity - more likely events contains less information, (ii) Non-negativity, (iii) Events with certainty do not provide information, and (iv) Additivity - Information from independent events are additive. Notice the mutual information is always weakly larger than 0 and is minimized when prior and posterior coincide.[3]

Equilibrium Definition – The equilibrium concept needs to be adapted to accommodate the uncertainty of workers regarding firms' wage postings. The natural concept is the *perfect Bayesian equilibrium* (hereafter, PBE):

Definition 1 (Symmetric Perfect Bayesian Equilibrium with Information Acquisition)

A ***Symmetric Perfect Bayesian Equilibrium*** *is a tuple* $\left\{G^e(\mathbf{w}), F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}^e_j(\mathbf{s})\}_j, w^e(z|\mathbf{z})\right\}$*:*

1. (Optimal Posting) $w^e(z|\mathbf{z})$ *maximizes the firm's profit given* $F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}^e_j(\mathbf{s})\}_j$ *and other firms use the same strategy, if the firm has productivity* z *and the entire productivity profile is realized at* $\mathbf{z}$*;*

2. (Optimal Search) $F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}^e_j(\mathbf{s})\}_j$ *maximizes every worker's payoff given belief* $G^e(\mathbf{w})$ *and other workers using the same strategy;*

3. (Consistency) $G^e(\mathbf{w})$ *is satisfies the Bayes rule given the productivity distribution* $G(\mathbf{z})$ *and* $w^e(z|\mathbf{z})$ *on the equilibrium path.*

In a PBE, workers form a belief regarding firms' equilibrium wage profiles according to Bayes rule. Workers optimally choose how to learn about the wage profile and how to apply for jobs based on their believes of the equilibrium wage profile. A subgame equilibrium is defined as a collection of learning and search strategy that maximize

[3] A result from Jensen's inequality.

every worker's payoff given other workers' learning and search strategy, as well as the belief. Firms take as given the outcomes of subgames and determine their optimal wage postings. I focus on a symmetric perfect Bayesian equilibrium equilibrium where (i) workers adopt identical learning and search strategy and (ii) firms adopt pure strategy (firms with the same productivity post the same wage).

Subgame Equilibrium – First characterize the subgame given any belief of wages $G^e(\mathbf{w})$.[4] A symmetric subgame equilibrium is $\left\{F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}_j^e(\mathbf{s})\}_j\right\}$ such that every worker finds it optimal to adopt the equilibrium strategy when other workers do the same. Mathematically, it requires $\left\{F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}_j^e(\mathbf{s})\}_j\right\}$ solves the following fixed-point problem:

$$\left\{F^e(\mathbf{s}|\mathbf{w}), \{\tilde{q}_j^e(\mathbf{s})\}_j\right\} = \underset{F(\mathbf{s}|\mathbf{w}),\{\tilde{q}_j(\mathbf{s})\}_j}{\arg\max} \int_{\mathbf{w}}\int_{\mathbf{s}}\sum_j \frac{1-(1-q_j^e)^I}{Iq_j^e} w_j \tilde{q}_j(\mathbf{s}) F(d\mathbf{s}|\mathbf{w}) G(d\mathbf{w})$$

$$- c\left(\mathbb{H}(G) - E_{\mathbf{s}}\mathbb{H}(F_{\mathbf{w}|\mathbf{s}})\right),$$

s.t.

$$\sum_j \tilde{q}_j(\mathbf{s}) = 1,$$

$$\int_{\mathbf{s}} F(d\mathbf{s}|\mathbf{w}) = 1$$

$$q_j^e(\mathbf{w}) = \int_s \tilde{q}_j^e(\mathbf{s}) F^e(d\mathbf{s}|\mathbf{w}).$$

Given other workers' use the equilibrium learning and search strategy $\{F^e(\mathbf{s}|\mathbf{w}), \tilde{q}_j^e(\mathbf{s})\}$, the probability of any individual worker applying to firm j is $q_j^e(\mathbf{w}) = \int_s q_j(\mathbf{s})F(d\mathbf{s}|\mathbf{w})$. This probability is crucial for the link between a perfect Bayesian equilibrium with rational inattention and the subgame perfect equilibrium with observed wage postings and costly directed search. First, it summarizes the probability that any other worker applying to the same firm and thus is sufficient for the calculation of the job finding probability at every firm[5]. Second, it has the interpretation of a recommendation signal.

[4] All workers face the same trading environment, so they form the same consistent belief about wage profile on the equilibrium path.

[5] Implicitly, I assume that workers do not receive correlated signals, so the law of large number holds.

Suppose I restrict the feasible set of signal structures to be signals that directly suggest whether to apply to firm j. Workers are free to choose the probability of these recommendation signals, with the restriction that the probabilities of recommendations add up to 1. Restricting attention to these recommendation signal is without loss of generality. In other words, any signal structure combined with optimal search decision can be represented as a recommendation signal directly telling workers where to apply that is always followed by workers. This result is similar to the logic of the revelation principle. In the setting of rational inattention, this result comes from two features of the learning and search decision: (1) The conditional distribution of signals and the search decision enter multiplicatively into the expected income and (2) the cost of acquiring information from Shannon's entropy has a chain rule.[6] This result has intuitive economic interpretation: workers do not gather new information in the search stage. The recommendation strategy carries the same amount of information content as the signal structure that induces workers to behave as the recommendation strategy.

Matějka and McKay (2015) states this result in a decision theory context. They show that a decision problem with uncertainty of payoffs and entropy cost in acquiring information can be solved as a decision problem with observed payoffs and a cost based on the mutual information between the recommendation signal and a baseline probability. The baseline probability is the search probability according to the prior. Lemma 1 restated their results in a frictional environment.

Lemma 1 (Matějka and McKay (2015) with Search Friction)
$\{F(\mathbf{s}|\mathbf{w}), \{\tilde{q}_j(\mathbf{s})\}_j\}$ *solve worker's problem given the equilibrium recommendation strategy* $q^e_j(\mathbf{w})$ *if and only if the recommendation strategy* $q_j(\mathbf{w}) = \int_{\mathbf{s}} \tilde{q}_j(\mathbf{s})F(d\mathbf{s}|\mathbf{w})d\mathbf{s}$ *solves the following problem*

$$\max_{\{q_j(\mathbf{w})\}} \int_{\mathbf{w}} \sum_j \frac{1-(1-q^e_j(\mathbf{w}))^I}{I q^e_j(\mathbf{w})} w_j q_j(\mathbf{w}) G(d\mathbf{w}) - c\left(\mathbb{H}(\bar{q}) - \int_{\mathbf{w}} \mathbb{H}(q(\mathbf{w}))dG(\mathbf{w})\right),$$

[6] For two random variables X and Y, conditional information of X on Y equals the mutual information of (X, Y) minus the information of Y.

s.t.

$$\sum_j q_j(\mathbf{w}) = 1,$$

$$\bar{q}_j = \int_{\mathbf{w}} q_j(\mathbf{w}) G(d\mathbf{w}).$$

Two assumptions in this paper further simplify this problem: (1) Firms draw productivities from the same distribution; (2) In a symmetric equilibrium, firms with the same productivity adopt an identical strategy. As a result, a priori firm identity does not reveal any information regarding wages in a symmetric equilibrium. Uniform distribution $\frac{1}{J}$ measures the information content in prior distribution. In another word, in a symmetric equilibrium, the benchmark probability is $\bar{q}_j = \frac{1}{J}$. In the context of search theory, this is the random search strategy. Reduction in uncertainty is measured by the difference of entropy from random search strategy and the actual strategies after information acquisition.

Lemma 1 is a powerful result. I can essentially reduce the strategy space of worker's problem from a high-dimensional space of distribution into the search probability among J firms, for a consistent belief of equilibrium wage distribution $G(\mathbf{w})$. Additionally, in a symmetric equilibrium with identical productivity distribution, I can analyze the subgames node-by-node, because the benchmark strategy $\bar{q}_j$ does not depend on the belief $G(\mathbf{w})$.

Corollary 0.1 (Subgame Recommendation Equilibrium)
$\{F(\mathbf{s}|\mathbf{w}), \{q_j(\mathbf{s})\}_j\}$ *is a subgame equilibrium given* $G(\mathbf{w})$ *in a symmetric perfect Bayesian equilibrium if and only if it solves the following fixed point problem*

$$q_j^e(\mathbf{w}) = \arg\max_{\{q_j(\mathbf{w})\}} \int_{\mathbf{w}} \sum_j \frac{1-(1-q_j^e(\mathbf{w}))^I}{I q_j^e(\mathbf{w})} w_j q_j(\mathbf{w}) G(d\mathbf{w}) - c \int_{\mathbf{w}} \sum_j q_j(\mathbf{w}) \log \frac{q_j(\mathbf{w})}{1/J} dG(\mathbf{w}),$$

s.t.

$$\sum_j q_j(\mathbf{w}) = 1$$

Proof: Take the results of 1; Write out the mutual information using $q_j(\mathbf{w})$ and benchmark $\frac{1}{J}$. *Q.E.D.*

The subgame recommendation equilibrium already looks very similar to the subgame equilibrium in an economy where workers observe the wages and pay a cost to direct search. One caveat: there is no restriction on the wage profiles that are with zero probability given $G(\mathbf{w})$. For wage profiles that are off-equilibrium, the definition of a perfect Bayesian equilibrium does not put any restriction on worker's belief. Because workers do not think these wage profiles are possible, they will never gather information about these non-existent wage profiles. Worker's search decision might be ill-informed if firms actually deviate to those wage profiles. The search decisions at those ill-informed states might prevent firms from actually deviating to those states. One could construct multiple equilibria using this logic.

Symmetric Perfect Recommendation Equilibrium – I adopt the equilibrium refinement as in Ravid (2019). Formally, it requires that for any wage profile, I can find some perturbation that visits this wage profile with positive probability and the subgame recommendation equilibrium is still an equilibrium given such a perturbation.

Definition 2 (Perfect Recommendation Equilibrium with Information Acquisition) *A symmetric **Perfect Recommendation Equilibrium** is a symmetric **Perfect Bayesian Equilibrium** such that*

1. (credible response) $q^e_j(\mathbf{w})$ is a credible response to $\{G^e(\mathbf{w}), w^e(z|\mathbf{z})\}$:

For every $\mathbf{w} = (w_1, ..., w_J)$, there exists a sequence $(G^n, \{\sigma^n_j\}_j)_n$ such that:

a. $\sigma^n_j(w_j|\mathbf{z}) > 0$ for every $\mathbf{z}$ and every $j-th$ element of $\mathbf{w}$;

b. $\sigma^n_j(w|\mathbf{z})$ converges strongly to $\delta_{w(z_j|\mathbf{z})}$ for every $\mathbf{z}$

c. G^n is consistent with $\{\sigma^n_j\}_j$;

d. For all n, $q^e_j(\mathbf{w})$ is the subgame recommendation equilibrium given G^n.

2. (Attentive) There exits $\mathbf{w}$ such that workers apply to every firm with positive probability.

Proposition 1

$q^e_j(\mathbf{w})$ is a credible response to $\{G^e(\mathbf{w}), w^e(z|\mathbf{z})\}$ if and only if for every fixed $\mathbf{w}$, $q^e_j(\mathbf{w})$ is the solution to the subgame equilibrium with costly directed search and observed wage $\mathbf{w}$.

PROOF: See Appendix. *Q.E.D.*

Firms face identical information environment in the game with observed wage profiles and the game with information acquisition. From proposition 9, I show the subgame equilibrium outcomes in the game with rational inattention and that in the game with observed wage profile and costly directed search are identical. Given the solution to the subgame equilibrium is unique, firms in the two games face identical problem. In conclusion, a symmetric perfect recommendation equilibrium can be solved as a collection of subgame perfect equilibrium with observed wage profiles and cost of directing search, given different productivity vectors.

Corollary 1.1 (Equivalence between symmetric perfect recommendation equilibrium and equilibrium with costly directed search)
$\{G^e(\mathbf{w}), q^e_j(\mathbf{w}), w^e(z|\mathbf{z})\}$ *is a symmetric recommendation equilibrium if an only if for every* $\mathbf{z}$*,* $(\{q^e_j(\mathbf{w})\}_j, w(z|\mathbf{z}))$ *is a symmetric subgame perfect equilibrium with full information and costly directed search.*

Corollary 1.2 (Existence of Symmetric perfect recommendation equilibrium)
A symmetric perfect recommendation equilibrium exits.

PROOF: Proposition 2 establishes that a symmetric subgame perfect equilibrium with costly directed search exists for any productivity $\mathbf{z}$. A symmetric subgame perfect equilibrium with full information and costly directed search is also a symmetric perfect recommendation equilibrium by corollary 9.1. *Q.E.D.*

2. ALTERNATIVE COST FUNCTIONS

So far, I focus on the K-L divergence as the cost of directing search, because of its foundation in information theory and its tractability. To show this framework is generalizable to other parametrization of cost functions, I now analyze the partially directed search for a general class of divergence measures called f-divergence. Specifically, the f-divergence takes the Radon-Nikodym derivative between the chosen search strategy and the uniform distribution a_j and evaluate the integral of the following form:

$$\text{Cost of Directing Search} = \int_0^1 \phi(a_j)dj$$

where ϕ is increasing and convex. To ensure non-deviation is costless, $\phi(1) = 0$. The f-divergence nests the K-L divergence as a special case when $\phi(a) = a \log a$. Now, I show partially directed search can also be motivated by the general f-divergence, the entropic competitive search equilibrium can be similarly defined, and the inefficiency of market equilibrium also exists. For simplicity, I assume $b = 0$ and $\mu = 1$.

Partially Directed Search with f-divergence Consider the worker's problem given any wage function ω and queue function $q(\omega)$. Workers' problem is as in equation (1). The worker's problem is again convex: it has a strictly concave objective function and linear constraint. The optimal search decision implies the search strategy must be a solution to condition (11). The marginal cost of applying to firm j is now measured by $c\phi'(q_j)$.

$$\max_a \int_0^1 m(q_j(\omega))\omega_j^+ q_j dj - c\int_0^1 \phi(q_j)dj \tag{1}$$

s.t.

$$\int_0^1 q_j dj = 1$$

$$m(q_j(\omega))\omega_j^+ - c\phi'(q_j) = V \tag{2}$$

Entropic Competitive Search Equilibrium with f-divergence The equilibrium with the general cost function can be accordingly defined: I look for wages that maximize firm's profit, search strategy that maximizes workers' payoff, and the equilibrium supply curve that is consistent with worker's decisions. I directly state the equivalent problem in equation Definition 6.

Definition 3 (Equivalent problem with f-divergence)

1. firm's optimality given V

$$\{q(V), w(V)\} = \max_{w,q} n(q)(z_j - w)$$

s.t.

$$m(q)w - c\phi'(q) = V$$

$$w \geq 0$$

2. Market Clearing

$$\int_0^1 q(V^e)dj = 1$$

Proving the existence of uniqueness follows the same logic as the case with K-L divergence: Firm's problem is strictly convex given any V. The optimal choice of queue is decreasing in V with at least a positive measure of firms that is strictly decreasing. Therefore, law of demand holds for the aggregate demand of applicants. However, the result on wage is different for the general class of cost function. Specifically, in equation (13), the markdown due to the cost of directing search depends on the curvature of cost function, which is zero when $\phi(a) = a \log a$.

$$\max\{n'(q_j^e)z_j - c\phi''(q_j^e), 0\} - c\phi(q_j^e) = V^e$$

$$w_j = \max\{\frac{n'(q_j)}{m(q_j)}z_j - c\frac{\phi''(q_j)}{m(q_j)}, 0\} \tag{3}$$

Efficiency with f-divergence The planner's problem is defined in equation (14). The constrained efficient allocation equalizes the benefit and cost of applying to firm j. Comparing this allocation the allocation from the market equilibrium, I find the markdown due to cost of directing search has impact on efficiency. The curvature in cost of directing search is crucial for how monopsony power distorts allocation to firms. With K-L divergence, all firms extract a constant markdown in expectation. In the general case, the markdown differs for firms with heterogeneous productivities. However, the inefficiency due to incentive compatibility constraint of workers prevails in the general case: In the equilibrium with general f-divergence, there can be cases where positive measure of firms post workers' outside option. Given this result, our discussion about minimum wage holds for general cost functions.

$$\max_{q_j} \int_0^1 n(q_j)z_j dj - c\int_0^1 \phi(q_j)dj \tag{4}$$

s.t.

$$\int_0^1 q_j dj = 1$$

$$q_j = q_j$$

$$n'(q_j^*) - c\phi'(q_j^*) = V^*$$

3. PARTIALLY DIRECTED SEARCH AND STANDARD MONOPSONY MODELS

This section rewrites the firms' problem in terms of expected hiring n instead of queue length. To do so serves two purpose. First, it makes clear the origins of market power in a partially directed search model. Second, this steps relate a partially directed search model to a standard model of monopsony. The partially directed search model provides an alternative interpretation of the labor supply elasticity estimated from these models. To prepare the notations, I define the the inverse function of the job-filling probability to be $g(n)$. This function maps the expected hirings into the number of application per firm:

$$g(N) = n^{-1}(N).$$

Let $W(n; V, c)$ be the wage a firm needs to pay if it plans to hire n workers when the market utility is V. Inverting the optimal decision of workers I get a labor supply curve:

$$W(n; V, c) = b + \frac{g(n)}{n}\Big(V + c\log g(n)\Big). \tag{5}$$

Given the market utility, firm j's problem is as in equation 5. Firm j decides on the number of workers to hire. Hiring n workers produces nz_j and incurs a total labor cost of $nW(n; V, c)$. A constraint is in place for the firm's problem. It reflects the fact workers always have the option to walk away from potential matches. So firms can never hire any one if they promise a wage below b. This formulation makes it clear the role of outside option in the partially directed search model: Workers' outside option is a potentially binding "minimum" wage.

$$\max_n \quad z_j n - nW(n; V, c),$$

s.t.

$$W(n; V, c) \geq b.$$

To analyze the firm's problem, I first derive the marginal cost curve $MC(n; V, c)$:

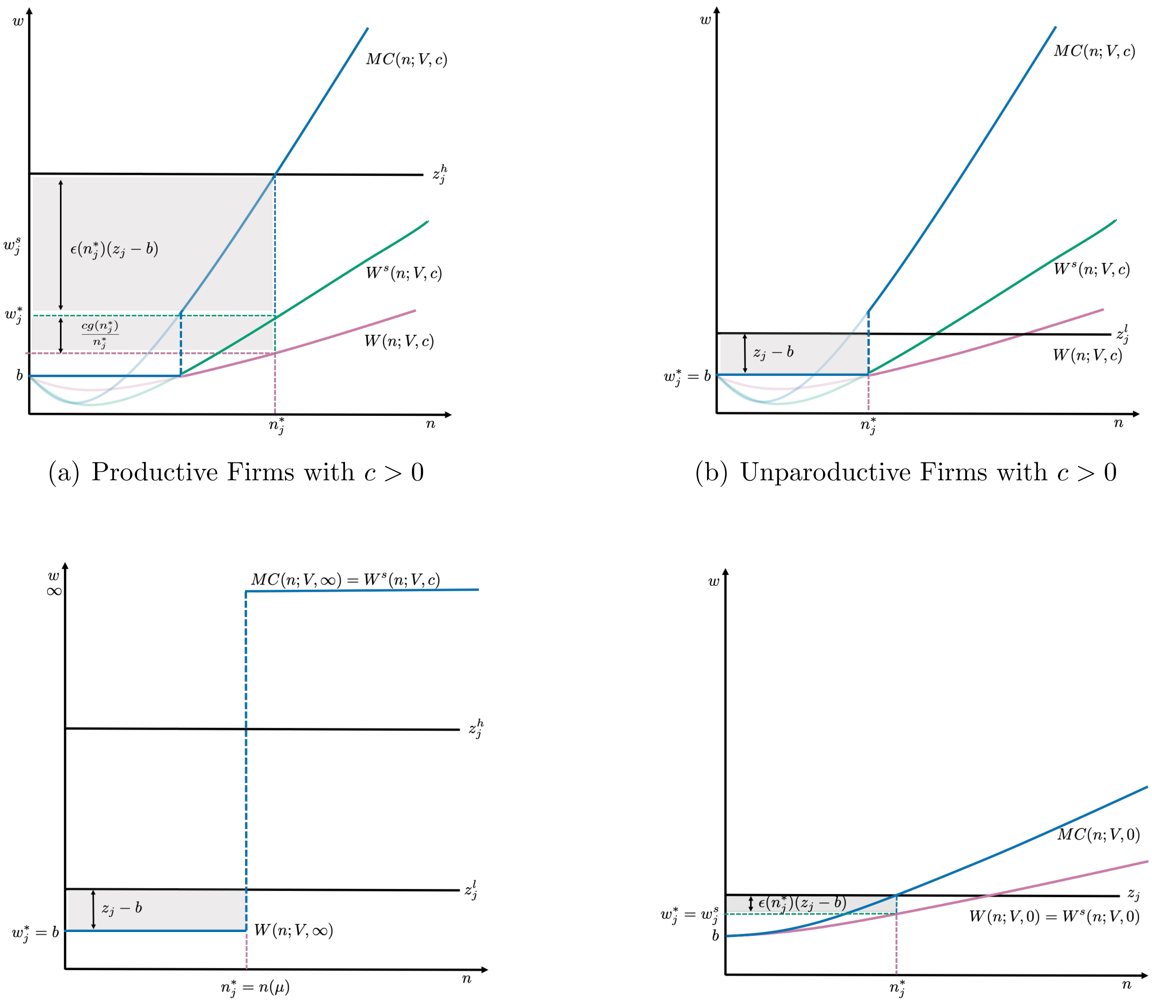
$$MC(n; V, c) = W(n; V, c) + nW'(n; V, c) = b + g'(n)\Big(V + c \log g(n)\Big) + g'(n)c$$

The marginal cost has two components, the first component reflects the contribution of a marginal worker to the matching process at firm j. In order to hire extra worker, the firm has to compensate workers their outside option b and their contribution to the matching process. Due to the cost in directing search, the red component reflects the market power due to cost of directing search. I will define $W^s(n; V, c)$ as the social value component:

$$W^s(n; V, c) = W(n; V, c) + nW'(n; V, c) = b + g'(n)\Big(V + c \log g(n)\Big).$$

Figure 5 plots the labor supply and labor demand curve given fixed V. Panel (a) is the case with a positive cost of directing search for a productive firm. The labor supply curve is in pink. Any wage below worker's outside option results in zero employment, so the labor supply curve is bounded below by b. For the employment that results in a wage above b, the labor supply curve is upward-sloping. (The segment below is not necessarily increasing due to the functional form of K-L divergence; however, it does not matter because of the constraint). The social value function of marginal worker W^s is in green and is above the labor supply curve, reflecting the search friction in the economy. To hire additional worker, the firm has to attract more than one applicants, which changes the matching probability of everyone applying to the same firm. The marginal cost of hiring additional worker, MC, is further above W^s reflecting the cost of directing search. There is a discontinuous jump of marginal cost curve. For the units of employment lower than threshold $AC = b$, all units must be hired at a constant cost (b), so there is no distortion between marginal and average cost. Beyond this threshold, the marginal cost jumps up because in order to hire more the firm offers a higher wage, for every job searcher in the market.

To find the optimal input of the firm, I look for the crossing point of marginal productivity z_j and the marginal cost of recruiting MC. The level of employment is



(a) Productive Firms with $c > 0$

(b) Unparoductive Firms with $c > 0$

(c) $c = \infty$

(d) $c = 0$

Figure 1: Supply Demand Diagram for Fixed V

optimal for firm j because the marginal benefit equals the marginal cost. The wage offered to workers is the wage required to hire n_j^* workers. The wage is below marginal product for two reasons: (1) the markdown due to search friction. Firms need to be compensated for creating jobs (the gap between MC curve and W^s curve); (2) the markdown due to the cost of directing search (the gap between W^s and W curve). It is evident that the markdown due to the oligopoly competition vanishes in the entropic competitive search equilibrium.

Panel (b) plots the input decision of an unproductive firm for the same cost of directing search. For this firm, productivity is low enough that the optimal employment is at the threshold $AC = b$. At this point, social value and wage function coincide, because there is the first unit that firm j decides to post higher wage in order to hire more. Firm j takes all the gains from trade and workers get paid their outside option.

Panel (c) plots the case $c \to \infty$. In this case, distorting search is infinitely costly. To hire more workers than the exogenous queue μ, the firm needs to post an infinite wage, which can never be optimal if $y_j < \infty$. In this case, both firms (productive and unproductive) offer workers' outside option, and extract all the gains from trade.

Panel (c) plots the case $c \to 0$. In this case, the labor supply curve and the social value curve always coincide. Firms and workers both get their contributions to the matching process. There is no bunching at workers' outside option because a relatively more productive firm can always differentiate itself from an unproductive firm. Because search is costless, posting a slightly higher wage attracts strictly more workers.

Proof of Proposition 9

I want to show that a symmetric perfect Bayesian equilibrium is indeed a Perfect Recommendation Equilibrium. To do so, I just need to check the two refinements are met.

Costly Directed Search to Rational Inattention

Want to show if $q^e(\mathbf{w})$ solves the following mapping for every $\mathbf{w}$, then it must be credible and attentive.

$$\{q_j^e\}_j = \arg\max_{q_j} \sum_j \frac{1-(1-q_j^e)^I}{Iq_j^e} w_j q_j - c \sum q_j \log \frac{q_j}{1/J} \tag{6}$$

s.t.

$$\sum_j q_j = 1$$

1. Credible Response

Consider a fixed $\mathbf{w}$, the goal is to construct a sequence of believes and firm strategy G^n, σ_j^n such that $\mathbf{w}$ is visited with positive probability. With these believes, the subgame equilibrium in a symmetric perfect Bayesian equilibrium is still subgame equilibrium with the new belief. Our strategy is to construct G_n such that the firms stay identical ex'ante. To do so: For any perturbation wage profile $\mathbf{w} \geq 0$, let $\delta_{\mathbf{w}}$ be the delta function with parameter $\mathbf{w}$. For any equilibrium wage profile $\mathbf{w}^e$, define the perturbation firm strategy:

$$\sigma_j^n = \left(1 - \frac{1}{n}\right) \delta_{w^e} + \frac{1}{n}\left(\sum_{k=1}^{J} \frac{1}{J} \delta_{\tilde{w}^k}\right)$$

where $\tilde{\mathbf{w}}^k$ is constructed by a shuffling of perturbation wage profile $\mathbf{w}$. Specifically, I can construct such perturbation by shuffling firm identities in the following matrix. Each column has the same element, but I sequentially shuffle to leading elements to the tail. Call the k-th column of following matrix the wage profiles $\tilde{w}^k$.

$$\begin{bmatrix} w_1 & w_2 & w_3 & \dots & w_J \\ w_2 & w_3 & w_4 & \dots & w_1 \\ \dots & \dots & \dots & \dots & \dots \\ w_J & w_1 & w_2 & \dots & w_{J-1} \end{bmatrix}$$

The potential issue with an arbitrary wage perturbation to $\mathbf{w}$ is that the perturbation to $\mathbf{w}$ might induce the benchmark strategy (the strategy before acquiring information) to be non-uniform. In such a case, using uniform search strategy as the benchmark strategy no longer characterize the optimal strategy. My solution is to find a collection counter-perturbations such that the benchmark strategy stays uniform. I construct such perturbation by shuffling firm identities. Given the wage profiles, the corresponding subgame equilibrium in the game with observed wages and cost of directing search must be the following matrix

$$\begin{bmatrix} q_1^e & q_2^e & q_3^e & \cdots & q_J^e \\ q_2^e & q_3^e & q_4^e & \cdots & q_1^e \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_J^e & q_1^e & q_2^e & \cdots & q_{J-1}^e \end{bmatrix}$$

No matter what is the perturbation profile $\mathbf{w}$, the row sum of the above matrix must be 1. So by uniformly weight the counter-perturbation profiles, I maintain a uniform benchmark strategy. This means equation (23) still characterize the subgame perfect equilibrium given the perturbated firm strategy and belief.

2. Attentive

For wage profile that equalize across all firms, it is always a subgame equilibrium to put positive weight on all firms. So the equilibrium outcome is attentive.

Rational Inattention to Costly Directed Search On the equilibrium path, a symmetric equilibrium means two firms use the same strategy if they have the same strategy. Because productivity distribution is identical across firms, this means the belief on wage profile must be identical across firms on the equilibrium path. Thus if any benchmark strategy is the equilibrium outcome, it must be uniform. For workers' strategy to be credible, corollary 8.1 implies the equation (23) must hold for every $\mathbf{w}$.

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