Assortative Matching with Private Information

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September 17, 2025

Abstract

We study matching markets with two-sided private information where agents care about their partner's unobserved type. Competing platforms post fees and promised partner distributions, while privately-informed agents direct their search to preferred platforms. Under common ranking and supermodularity, we prove that equilibrium must be separating: each platform attracts exactly one type from each side. We characterize positive assortative matching equilibria and provide sufficient conditions for their existence. The mechanism for separation depends crucially on willingness-to-pay: when more desirable types have higher willingness-to-pay for any match, they separate through high fees; when they have lower willingness-to-pay, they separate through low matching rates. We find that equilibrium may require a novel matching, mixing positive and negative sorting. The negative sorting raises the value of the least desirable agents, alleviating the cost of private information.

*We thank the seminar participants at Carnegie Mellon Tepper, NBER Summer Institute, Peking University, Pennsylvania State University, the University of Edinburgh, University of Toronto, and Univerisity of Zurich for their comments and discussions.

1 Introduction

This paper develops a model of markets with two-sided private information where heterogeneous agents care directly about their trading partner's unobserved type. Platforms compete to match agents, but cannot observe the agents' type. Instead, agents sort by selecting the platform that maximizes their expected utility. Our framework aims to provide a foundation for studying markets ranging from online dating and job search platforms to peer-to-peer exchanges where both sides have private information about attributes that matter to their potential partners.

The key innovation is a competitive search model where platforms post terms-of-trade for two sides of a market: binding fee commitments and promised distributions of trading partners. While platforms can commit to fees, promised distributions are cheap talk because agents' types are private information. Yet in equilibrium these promises must be kept. For example, if a platform promises to attract high-quality types to one side of the market but charges fees that only low-quality types find worthwhile, the promise is not credible and no one will try to trade at this terms-of-trade.

Unlike mechanism design problems where agents care only about transfers, here agents care about whom they match with. A worker cares about job quality, while the firm cares about worker skill. Dating app users care about the character of potential matches. In markets for expertise, buyers care about sellers' knowledge about the quality of what they are selling, while sellers care about buyers' ability to assess quality. This direct preference over partner types, combined with two-sided private information, fundamentally shapes how markets operate.

Our main results hold under two natural assumptions: first, that all agents agree on who is desirable (Common Ranking), and second, that more desirable agents have a stronger preference for more desirable partners (Supermodularity). Under these conditions, we prove that any terms-of-trade that maximizes platform profits must attract exactly one type from each side of the market (Corollary 1). Moreover, only downward incentive constraints bind: a platform may need to distort the terms-of-trade in order to thwart lower agent types' desire to participate, but in equilibrium higher agent types are not tempted to come to the platform (Corollary 2). These results emerge from competitive pressure between platforms. Competition determines everyone's outside option, the value of matching using a different platform.

Much of our analysis focuses on positive assortative matching, where more desirable types match together. This requires that observationally equivalent agents choose different terms-of-trade, depending on their private information. How this occurs depends crucially on whether agents have increasing or decreasing willingness-to-pay for matches. With increasing willingness-to-pay, where higher types value any given match more than lower types, platforms separate agents by charging different fees. For instance, in labor markets, a high-skill worker may value matching with any given firm more than a low-skill worker does. Higher types pay more to access platforms that attract better partners, while all types match at the maximum feasible rate (Proposition 2). The intuition is straightforward: since higher types value matching more, they can be separated by charging fees that lower types find prohibitively expensive relative to their valuation.

With decreasing willingness-to-pay, where higher types value any given match less than lower types, separation works through matching probabilities. In disease transmission models, healthy individuals value any given match less than sick individuals do, because matching comes with a risk of becoming sick. Higher (healthier) types match less frequently to avoid undesirable partners, while fees are zero (Proposition 3). Here, reducing the matching rate is an effective screening device because the opportunity cost of not matching is greater for lower types who value each match more highly.

When platforms face positive advertising costs, both mechanisms operate simultaneously but in opposite directions depending on willingness-to-pay (Proposition 4). With increasing willingness-to-pay, higher types pay higher fees and match more frequently. With decreasing willingness-to-pay, higher types pay lower fees and match less frequently. These opposing patterns reflect how platforms must cover their costs: with increasing willingness-to-pay, platforms can charge high-value types more and so are able to provide high matching rates. With decreasing willingness-to-pay, high types accept low matching rates to avoid bad partners. Since this is cheaper to provide, competition between platforms drives down fees.

Our final theoretical insight regards cross-subsidization in a separating equilibrium. It is well-known that the equilibrium of competitive markets with private information can be Pareto inefficient because cream-skimming prevents markets from cross-subsidizing less desirable agents (Guerrieri, Shimer and Wright, 2010). That is, the need to exclude a small number of undesirable agents from a market for desirable agents may reduce the efficiency of the market for desirable agents. The more desirable agents would collectively be willing to cross-subsidize the undesirable agents, but competition and free-riding prevents that from happening in equilibrium. Our paper shows how competition between platforms may alleviate that inefficiency when there is a sorting problem.

Under quite general conditions, we show that there is no equilibrium with positive assortative matching. The issue lies exclusively with the lowest type of agent, who face no binding incentive constraint, making them particularly desirable partners. Because of this, platforms create a market that matches this type of agent with a higher type. This raises the utility of the lowest type above what they would get in the absence of private information. That is, the lowest type gets an information rent. Importantly, the information rent relaxes incentive constraints for everyone else, raising the utility of every type of agent. Equilibrium then combines a region of negative assortative matching for the lowest types, who are cross-subsidized, with positive assortative matching for all higher types. A non-trivial interval of types engage in both types of matches. We believe that this insight is new: competition between platforms can lead to cross-subsidization in a separating equilibrium by changing the equilibrium matching pattern.

Our results illuminate how real-world matching markets successfully separate types despite asymmetric information. In practice, we see both fees and matching frequencies utilized. Building on early insights of Veblen (1900), research by Spence (1973), Pesendorfer (1995), and Bagwell and Bernheim (1996) showed how wasteful expenditures can signal an individual's unobserved qualities. Conversely, research by Guerrieri and Shimer (2014) shows that illiquidity can serve a similar function, with the owners of high quality assets setting a high price that attracts few buyers. This separates them from the owners of lower quality assets because the high-quality owners are more willing to accept the risk of not selling the asset.

Our theory offers a prediction about when we would expect to see wasteful expenditures (high fees paid to platforms) and when we would expect to see illiquidity (low matching probabilities) in real-world markets. The method of signaling depends on whether more desirable types are more or less eager to match with any given partner, compared to the alternative of not matching. Dating markets provide a particularly vivid illustration of these mechanisms at work.

Consider Tinder's experiments with premium tiers. For years, Tinder displayed a gold badge for premium subscribers, but recently removed this visible indicator. Our theory suggests an interpretation of this decision. If in equilibrium premium subscribers are more desirable to other users, perhaps because they are more serious about finding partners, then the gold badge serves as a valuable signal worth paying for. But if premium subscribers are less desirable on average, perhaps subscribing because they struggle to find matches using the free tier, then the badge becomes a stigma. Removing the visible badge would then increase premium subscriptions by eliminating this negative signal. Even more dramatically, Tinder briefly offered a "Select" tier costing $499 per month with minimal additional features. Such an extreme price could either signal wealth and serious intent (making the badge valuable) or desperation (making it toxic). The tier's quick discontinuation suggests the latter interpretation prevailed.

In this second scenario where premium users have a higher willingness-to-pay but are less desirable, our theory makes an additional prediction: separation should occur through more desirable users choosing not to pay for Tinder Gold and consequently receiving fewer matching opportunities, i.e. limited daily swipes and less frequent appearances in others' swipe decks. Less desirable users with higher willingness-to-pay would subscribe to premium tiers and receive more matching opportunities through unlimited swiping and boosted visibility. While more desirable users likely still achieve more actual matches despite fewer opportunities, the platform separates types by controlling access to the matching process itself. This illustrates how platforms may use multiple instruments, pricing and matching frequencies, to facilitate sorting when types are unobservable.

Related Literature

At a broad level, our paper explores the connection between adverse selection (Akerlof, 1970) and assortative matching (Becker, 1973) in competitive markets. Our paper contributes to several strands of the literature. Most directly, we extend the competitive search framework to environments with two-sided private information where agents care about partner types. Early competitive search models had neither heterogeneity nor private information (Moen, 1997; Shimer, 1996). Subsequent research has extended the basic model to allow for two-sided heterogeneity in order to study assortative matching (Shi, 2001; Shimer, 2005; Eeckhout and Kircher, 2010), but still assumes that there is no relevant private information.[1]

Guerrieri, Shimer and Wright (2010) and Guerrieri and Shimer (2014) introduced private information to competitive search but maintained one-sided private information.[2] For example, workers have private information but firms do not, or buyers have private information about their valuation but sellers care only about price. The crucial difference in our environment is that both sides have private information and both sides care about their partner's type. This creates a fundamentally different screening problem. In Guerrieri, Shimer and Wright (2010), separation occurs when the uninformed party posts a contract to attract a particular type of informed party. In our model, both sides simultaneously screen and are screened, creating a fixed-point problem in beliefs about who participates in each market.

The literature on matching with private information has taken different approaches. Hoppe, Moldovanu and Sela (2009) study assortative matching when agents send costly signals, assuming positive assortative matching in signals. Their single-crossing conditions ensure higher types send higher signals in equilibrium. Damiano and Li (2007) and Hoppe, Moldovanu and Ozdenoren (2011) analyze monopolistic platforms that screen privately in-

[1] In Eeckhout and Kircher (2010), buyers are privately informed about their type, but sellers do not care about the buyers type, and so this has no impact on the equilibrium allocation or prices.

[2] There are other works that study environments with one-sided private information and search frictions, for example, Kim and Kircher (2015); Auster and Gottardi (2019); Albrecht et al. (2024); Auster et al. (2025).

formed agents and Damiano and Li (2008) consider a screening model with duopolistic platforms. Our paper differs in three key ways. First, we develop the natural benchmark of competitive markets, which focuses on how agents gain from trade, rather than how monopolists extract profits. Second, we endogenize both fees and matching rates as screening instruments, showing that separation can work through either channel. Third, we allow general payoff functions, showing that increasing versus decreasing willingness-to-pay fundamentally changes how markets operate.

Myerson and Satterthwaite (1983) provides the canonical model of bilateral trade with private information, but they assume that there is a single buyer and a single seller, sidestepping the matching problem. Moreover, each agent cares only about prices and their own private information, not directly about their trading partner's characteristics. In contrast, we focus on environments with many buyers and sellers where each agent cares both about prices (fees) and about whom they trade with.

Our technical approach builds on the mechanism design literature with type-dependent outside options. Lewis and Sappington (1989) show that countervailing incentives arise when types differ in their outside options, potentially causing both upward and downward incentive constraints to bind. Jullien (2000) characterizes optimal contracts with type-dependent participation constraints. In our setting, competition among platforms endogenously determines outside options (market utilities), and we show this competition ensures only downward constraints bind. This helps keep our model tractable, particularly for the analysis of positive assortative matching.

The paper also connects to the two-sided markets literature initiated by Rochet and Tirole (2003), Rochet and Tirole (2006) and Armstrong (2006). While that literature focuses on network effects and platform pricing with observable characteristics, we analyze how platforms facilitate matching when characteristics are unobservable. Weyl (2010) studies monopoly platform design with heterogeneous users, but maintains the assumption that users care only about the number, not the types, of users on the other side.

Our distinction between increasing and decreasing willingness-to-pay connects to the broader literature of private information and screening, since Akerlof (1970). In insurance markets, Rothschild and Stiglitz (1976) show that high-risk individuals have higher willingness-to-pay for any given coverage level, leading to positive correlation between coverage and premiums. Einav and Finkelstein (2011) survey evidence of "advantageous selection," where low-risk types have higher willingness-to-pay for any given coverage level, reversing standard predictions. Our model shows how both cases arise naturally depending on how own type affects the marginal utility of matching.

Roadmap

Section 2 presents the model, introducing platforms that facilitate matching between privately informed agents. Section 3 provides motivating examples including marriage markets, labor markets, expertise markets, and disease transmission. Section 4 establishes our main theoretical results on separation and the nature of binding incentive constraints. In Sections 5 and 6, we study the increasing and decreasing willingness-to-pay cases in detail when advertising is free and the economic environment is symmetric. We introduce our findings on cross-subsidization in these sections. Section 7 extends the analysis to a positive advertising cost. Section 8 concludes. Proofs are in the Appendix, along with a discussion of asymmetric environments and negative assortative matching.

2 Model

2.1 Platforms and Agents

We consider a static model of a two-sided market with three sets of risk-neutral participants: a-side agents, b-side agents, and platforms. We let $s \in \{a, b\}$ denote one side of the market and $\bar{s}$ denote the other side, so if $s = a$, $\bar{s} = b$ and vice versa.

There exists a fixed measure I^s of s-side agents. Each agent has private information about their type $i \in \mathbb{I}^s$, a compact subset of the unit interval. The distribution of types on side s is governed by an exogenous cumulative distribution function $F^s : \mathbb{I}^s \to [0, 1]$.

When a type-i, side-s agent matches with a type-j, side-$\bar{s}$, i receives a payoff $u^s(i, j)$ before paying any platform fees. When an agent fails to match, their payoff is normalized to zero. We impose the following key assumption on preferences:

Assumption 1 (Common Ranking) *Take any s, $i \in \mathbb{I}^s$, and $j, j' \in \mathbb{I}^{\bar{s}}$. If $j > j'$, $u^s(i, j) > u^s(i, j')$.*

This assumption requires that all agents on a given side of the market agree on the ranking of potential partners from the other side. That is, if any type-i, side-s agent prefers to match with j rather than j', then all other side-s agents share this preference. Given this commonality in preferences, we adopt the convention of labeling types such that a higher index indicates a partner who gives a higher payoff. Thus, $j > j'$ if and only if all agents on the other side of the market prefer to match with j to j'. Finally, we assume that u^s is continuous in its first argument.

There is also a fixed measure 1 of homogeneous platforms, each of which chooses its advertising effort at constant unit cost $c \geq 0$. This advertising effort facilitates matching between agents on the two sides of the market, with costs covered through endogenous fees paid by the agents. In our leading case, advertising effort is free, $c = 0$, but we also consider the possibility that advertising effort is costly, $c > 0$.

2.2 Terms-of-trade and Payoffs

A terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b}$ specifies four objects: the fees ϕ^a and ϕ^b charged to matched agents on each side and the distributions G^a and G^b of agent types participating on each side. We allow either ϕ^a or ϕ^b to be negative but assume $\phi^a + \phi^b \geq 0$. We denote by $\mathbb{T}$ the set of feasible terms-of-trade, consisting of all tuples where ϕ^a and ϕ^b are real numbers with nonnegative sum, and G^a and G^b are probability distributions with support on subsets of $\mathbb{I}^a$ and $\mathbb{I}^b$, respectively.

Agents direct their search toward particular terms-of-trade $\tau \in \mathbb{T}$. If a type-i, side-s agent succeeds in trading at terms-of-trade $\tau \in \mathbb{T}$ with probability $\lambda^s \in [0, 1]$, their expected payoff is

$$\bar{U}^s(i, \tau, \lambda^s) \equiv \lambda^s \left(\int_{\mathbb{I}^{\bar{s}}} u^s(i, j) dG^{\bar{s}}(j) - \phi^s \right). \tag{1}$$

That is, with probability λ^s the agent trades, receiving an expected gross payoff equal to their expected utility $u^s(i, j)$ across potential trading partners drawn from the distribution of agents on the other side of the market, $G^{\bar{s}}$, less the platform fee ϕ^s.

Platforms facilitate these trades. For a platform advertising terms-of-trade $\tau \in \mathbb{T}$ with associated matching probabilities λ^s, the expected gross profit per unit of advertising effort is

$$V(\tau, \lambda^a, \lambda^b) \equiv m(\lambda^a, \lambda^b)(\phi^a + \phi^b) \tag{2}$$

where $m(\lambda^a, \lambda^b)$ denotes the platform's matching probability as a function of the agents' matching probability, discussed in the next subsection. Each unit of advertising effort at terms-of-trade τ results in matches with probability $m(\lambda^a, \lambda^b)$, generating revenue $\phi^a + \phi^b$. The platform's net profit subtracts the advertising cost c from this gross profit.

2.3 Matching Function and Platform Matching Probability

To understand the platform matching probability m, we start with a more familiar object, the matching function M. We assume that in each market, the number of matches is a constant returns to scale function of the platform's advertising effort, the number of a-side agents, and the number of b side agents. Let α denote the advertising effort and $n^s \geq 0$ denote the number of s-side agents per unit of advertising effort at some terms-of-trade τ. Then the number of matches is $\alpha M(n^a, n^b)$, where $M : \mathbb{R}^2_+ \to \mathbb{R}_+$. Consistent with the constant returns to scale assumption, M is nondecreasing and strictly concave in (n^a, n^b), with $M(n^a, n^b) \leq \min\{n^a, n^b\}$. We also assume M is differentiable.

Agents' matching probability is related to the matching function via

$$\lambda^s = L^s(n^a, n^b) \equiv \frac{M(n^a, n^b)}{n^s} \leq 1. \tag{3}$$

for $s = a, b$ and $n^s > 0$. The numerator is the matching probability per unit of advertising effort, while the denominator is s-side agents per unit of advertising effort, and so the ratio is matching probability per s-side agent. This is well-defined when n^a and n^b are both strictly positive and one can handle the cases where n^a and/or n^b converge to either zero or infinity through appropriate limits.

Using this, we can express the platform matching probability via the identity

$$m(L^a(n^a, n^b), L^b(n^a, n^b)) \equiv M(n^a, n^b).$$

This is the probability that the platform matches as a function of the agents' matching probabilities, rather than as a function of the agent-advertising ratios. It is this object that we use in equation (2).

By varying n^a and n^b, we trace out a set of feasible agent matching probabilities:

$$\mathbb{\Lambda} \equiv \{(L^a(n^a, n^b), L^b(n^a, n^b)) | n^a, n^b \geq 0\} \subset [0, 1]^2.$$

The platform matching probability is a function $m : \mathbb{\Lambda} \to [0, 1]$. From the assumptions on the matching function, we get that $\mathbb{\Lambda}$ is a down set: if $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$, then $(\hat{\lambda}^a, \hat{\lambda}^b) \in \mathbb{\Lambda}$ for all $\hat{\lambda}^a \in [0, \lambda^a]$ and $\hat{\lambda}^b \in [0, \lambda^b]$. We also define

$$\mathbb{\Lambda}^o \equiv \{(\lambda^a, \lambda^b) \in \mathbb{\Lambda} | m(\lambda^a, \lambda^b) > 0\} \subset \mathbb{\Lambda}.$$

This excludes from the set $\mathbb{\Lambda}$ the set of points (λ^a, λ^b) where $m(\lambda^a, \lambda^b) = 0$.

We can also go back from the platform matching probability to the matching dfunction by first finding the number of s-side agents per unit of advertising effort. For all $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$,

$$n^s = N^s(\lambda^a, \lambda^b) \equiv \frac{m(\lambda^a, \lambda^b)}{\lambda^s}. \tag{4}$$

We then recover the matching function via the identity

$$M(N^a(\lambda^a, \lambda^b), N^b(\lambda^a, \lambda^b)) = m(\lambda^a, \lambda^b).$$

Parametric Example. A parametric example illustrates these properties and clarifies the domain restriction in $\mathbb{\Lambda}$. Suppose M takes a CES form:

$$M(n^a, n^b) = (1 + (n^a)^{-\gamma} + (n^b)^{-\gamma})^{-\frac{1}{\gamma}}, \tag{5}$$

where $\gamma > 0$ and $1/(1+\gamma)$ is the elasticity of substitution between n^a and n^b. This matching function is continuous, increasing, strictly concave, and satisfies $M(n^a, n^b) \leq \min\{n^a, n^b\}$ for all $(n^a, n^b) \in \mathbb{R}^2_+$.

To express this in terms of matching probabilities, we first write equation (3) as

$$\lambda^s = L^s(n^a, n^b) = \left(\frac{1 + (n^a)^{-\gamma} + (n^b)^{-\gamma}}{(n^s)^{-\gamma}}\right)^{-\frac{1}{\gamma}}.$$

Since γ is positive, λ^s lies between 0 and 1. Inverting this gives us

$$n^s = N^s(\lambda^a, \lambda^b) = \left(\frac{1 - (\lambda^a)^{\gamma} - (\lambda^b)^{\gamma}}{(\lambda^s)^{\gamma}}\right)^{\frac{1}{\gamma}}.$$

Substituting back into equation (5) gives

$$m(\lambda^a, \lambda^b) = (1 - (\lambda^a)^{\gamma} - (\lambda^b)^{\gamma})^{\frac{1}{\gamma}}. \tag{6}$$

We can equally go from the platform matching probability (6) back to the matching function (5).

By varying (n^a, n^b) in the positive real quadrant, we find that the feasible domain for matching probabilities is

$$\mathbb{\Lambda} = \{(\lambda^a, \lambda^b) | (\lambda^a, \lambda^b) \geq 0 \text{ and } (\lambda^a)^{\gamma} + (\lambda^b)^{\gamma} \leq 1\},$$

with $\mathbb{\Lambda}^o = \{(\lambda^a, \lambda^b) | (\lambda^a, \lambda^b) \geq 0 \text{ and } (\lambda^a)^{\gamma} + (\lambda^b)^{\gamma} < 1\}$. Thus the matching probabilities have a bounded domain $\mathbb{\Lambda}$ that is down set.[3]

2.4 Equilibrium

Our equilibrium concept builds on the competitive search literature pioneered by Moen (1997) and extended to environments with private information by Guerrieri, Shimer and

[3] We obtain points where $\lambda^s = 0 < \lambda^{\bar{s}}$ by taking the limit as $n^s \to \infty$ for fixed $n^{\bar{s}}$. Points with $\lambda^a = \lambda^b = 0$ correspond to the limit as n^a and n^b both grow without bound. We get points with $(\lambda^a)^{\gamma} + (\lambda^b)^{\gamma} = 1$ by taking the limit as n^a and n^b converge to zero with a fixed ratio.

Wright (2010). In these models, market-makers post terms-of-trade, and agents on both sides of the market direct their search toward their preferred terms. The resulting equilibrium combines price posting with rational expectations about matching probabilities.

The key innovation of the competitive search approach, relative to random search models, is that it allows market participants to use posted terms-of-trade to solve selection and incentive problems. As in Moen (1997), platforms in our setting act as market-makers who compete to attract agents. However, our setting differs in two important ways. First, following Guerrieri, Shimer and Wright (2010), we allow for private information about agent types. Second, platforms must simultaneously attract agents from both sides of the market, making this a two-sided matching problem with private information.

We break the definition of equilibrium into two parts. First, we define a partial equilibrium where everyone takes as given the agents' *market utility* $U^s(i)$, the utility that a type-i, side-s agent can obtain at their best possible terms-of-trade. The partial equilibrium concept requires that platforms cannot profitably deviate by offering a different terms-of-trade, taking these market utilities as given.

In the second step, we impose conditions that endogenously determine these market utilities. The market clearing conditions ensure that the total measure of each type choosing to trade equals the available supply of that type, while free entry ensures that platforms earn zero profits.

More formally, we begin with the partial equilibrium definition:

Definition 1 *A **partial equilibrium** $\{T^p, T, (\Lambda^s, U^s)_{s=a,b}\}$ is two sets $T \subseteq T^p \subseteq \mathbb{T}$ as well as functions $\Lambda^s : T^p \to \mathbb{\Lambda}$ and $U^s : \mathbb{I}^s \to \mathbb{R}+$ such that:*

1. *(Optimal Search) For all $\tau = (\phi^s, G^s)_{s=a,b} \in T^p$ and $s \in a, b$,*

   (a) $U^s(i) \geq \bar{U}^s(i, \tau, \Lambda^s(\tau))$ for all $i \in \mathbb{I}^s$;

   (b) $\int_{\mathbb{I}^s} U^s(i)dG^s(i) = \int_{\mathbb{I}^s} \bar{U}^s(i, \tau, \Lambda^s(\tau))dG^s(i)$;

2. *(Impossible Terms-of-Trade) For all $\tau = (\phi^s, G^s)_{s=a,b} \notin T^p$, there is no $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ such that*

   (a) $U^s(i) \geq \bar{U}^s(i, \tau, \lambda^s)$ for all $s \in \{a, b\}$ and $i \in \mathbb{I}^s$; and

   (b) $\int_{\mathbb{I}^s} U^s(i)dG^s(i) = \int_{\mathbb{I}^s} \bar{U}^s(i, \tau, \lambda^s)dG^s(i)$ for all $s \in \{a, b\}$;

3. *(Profit Maximization) $T = \arg\max_{\tau \in T^p} V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$.*

Let us first understand why the conditions in part 1 (Optimal Search) capture reasonable restrictions on behavior. Consider condition 1(a), which requires $U^s(i) \geq \bar{U}^s(i, \tau, \Lambda^s(\tau))$ for all agent types. Suppose this inequality were violated for some side s, type-i agent. Then these agents would strictly prefer this terms-of-trade to their market utility $U^s(i)$, causing them to flock to this terms-of-trade. This influx would drive down the matching probability $\Lambda^s(\tau)$ until the inequality is restored. Just as in traditional competitive search models, entry continues until agents are indifferent between this terms-of-trade and their market utility. Thus, any equilibrium must satisfy condition 1(a).

Condition 1(b) requires that agents who are supposed to come to the terms-of-trade (those in the support of G^s) obtain exactly their market utility. To see why this is necessary, suppose the expected utility were strictly less than market utility for some types in the support of G^s. Then these agents would prefer to search elsewhere, making it impossible for the terms-of-trade to yield the promised type distribution G^s. Conversely, condition 1(a) tells us that expected utility cannot exceed market utility. Thus, 1(b) is required for the promised type distribution to be consistent with agents' directed search incentives.

The second component of the definition identifies terms-of-trade that are impossible given market utilities. For instance, some terms-of-trade involve fees so high that no agent would be willing to pay them regardless of the matching probabilities. Alternatively, the promised type distribution G^s might be inconsistent with agents' incentives. For example, it may promise that only high types will participate when in fact only low types would be willing to do so. The impossible terms-of-trade condition formalizes this by requiring that for any $\tau \notin T^p$, there exist no feasible matching probabilities, $(\lambda^a, \lambda^b) \in \mathbb{A}$ that could make these terms-of-trade satisfy condition 1.

The third component determines which possible terms-of-trade at a given level of market utility. Given the matching probabilities $\Lambda^s(\tau)$ for each possible terms-of-trade, platforms choose terms-of-trade $T \subseteq T^p$ that maximize their profit per unit of advertising. This optimization reflects platforms' optimal behavior given agents' responses encoded in the matching probabilities.

This definition extends the competitive search framework to accommodate two-sided matching with private information. The matching probabilities $\Lambda^s(\tau)$ play a role analogous to market tightness in traditional competitive search models, but now must simultaneously clear both sides of the market while respecting agents' incentives. The partition into possible and impossible terms-of-trade provides a tractable way to incorporate these incentive constraints into the equilibrium concept.

A partial equilibrium determines which terms-of-trade could emerge given fixed market utilities $U^s(i)$. A competitive search equilibrium (CSE) endogenously determines these market utilities through free entry of platforms and market clearing.

Definition 2 *A* ***competitive search equilibrium*** *is a partial equilibrium* $\{T^p, T, (\Lambda^s, U^s)_{s=a,b}\}$ *and a measure* μ *on the set of profit-maximizing terms-of-trade* T *such that:*

1. *(Free Entry)* $c = V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$ *for all* $\tau \in T$*;*
2. *(Market Clearing)* $\forall s \in \{a, b\}$ *and* $I \subseteq \mathbb{I}^s$*,*

$$\int_I dF^s(i) I^s \geq \int_T \int_I N^s(\Lambda^a(\tau), \Lambda^b(\tau)) dG^s(i) d\mu(\tau)$$

with equality if $U^s(i) > 0$ *for all* $i \in I$*.*

3. *(Market Utility) For all* $s \in \{a, b\}$ *and* $i \in \mathbb{I}^s$ *with* $U^s(i) > 0$*,* $U^s(i) = \max_{\tau \in T} \bar{U}^s(i, \tau, \Lambda^s(\tau))$*.*

Let us examine each component of this definition. First, we introduce a measure μ over the set of profit-maximizing terms-of-trade T. This measure captures both the intensive margin (how much advertising there is at each terms-of-trade) and extensive margin (which terms-of-trade are offered). This formulation allows for a rich set of equilibrium outcomes, in particular ones where different platforms attract different distributions of agents at different terms-of-trade.

The free entry condition ensures that in equilibrium, no platform can earn positive profits net of the advertising cost c. If profits were positive at some terms-of-trade $\tau \in T$, more platforms would enter, offering these terms until the profit is competed away. Conversely, if profits were negative, platforms would exit until profits rise to zero or that terms-of-trade disappears entirely.

The market clearing condition equates the supply and demand for each set of agents. For any measurable set of types I, the left side represents the total supply of these types (I^s is the total measure of s-side agents). The right side integrates across all active terms-of-trade to find the total measure of these types being used, accounting for the matching probabilities $\Lambda^s(\tau)$ and promised type distributions G^s.

The weak inequality in market clearing allows for the possibility that some agent types earn zero market utility and do not participate in any terms-of-trade. However, if all types in set I earn strictly positive market utility, then supply must exactly equal demand. This reflects that agents cannot be excluded from terms-of-trade offering them strictly positive utility in equilibrium.

Finally, market clearing does not impose any restrictions on sets with zero measure. That is, market clearing only needs to hold for almost every type of agent with $U^s(i) > 0$. The last condition fills this gap, imposing that every agent who has positive market utility must get that level of utility in some terms-of-trade that gives platforms zero profits, i.e. at some $\tau \in T$.

2.5 Separation and Assortative Matching

We focus on two distinct characteristics of matching patterns that can emerge in equilibrium. The first concerns whether different types mix together on the same side of a single market, while the second concerns how types match across different markets. These patterns are important for characterizing equilibrium in two-sided markets.

First, we ask whether the equilibrium involves different types participating on the same side of a market, or whether types always separate into distinct markets. This motivates our definition of separation:

Definition 3 (Separation) *A terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b} \in \mathbb{T}$ is separating if for $s \in \{a, b\}$ there exists an i^s such that $dG^s(i^s) = 1$. A (partial or competitive search) equilibrium is separating if every profit-maximizing terms-of-trade $\tau \in T$ is separating.*

In a separating equilibrium, each active market attracts exactly one type from each side. This means that in equilibrium, an agent knows exactly who they will match with when they go to any active markets. This contrasts with pooling outcomes where multiple types might search in the same market.

Second, we examine whether there is systematic matching between agent types across markets. The classical notion of assortative matching describes whether high types match with other high types. In our setting with multiple markets, we need a definition that works across markets:

Definition 4 (Positive Assortative Matching) *Take any subset of terms-of-trade $\hat{T} \subset \mathbb{T}$. Select two elements $(\phi^s_k, G^s_k)_{s=a,b} \in \hat{T}$ for $k \in 1, 2$ and numbers i_k in the support of G^a_k and j_k in the support of G^b_k. If any such numbers satisfies $(i_1 - i_2)(j_1 - j_2) \geq 0$, then $\hat{T}$ has positive assortative matching (PAM). A (partial or competitive search) equilibrium has PAM if T has PAM.*

To understand this definition, consider two markets and pick any type i_1 that participates on the a-side of market 1 and any type i_2 that participates on the a-side of market 2. Similarly, pick types j_1 and j_2 from the b-sides of these markets. PAM requires that if $i_1 > i_2$, then $j_1 \geq j_2$. That is, if we find a higher a-side type in market 1 than in market 2, we must also find a weakly higher b-side type there. This captures the idea that higher types match together, but extends it to environments with multiple markets and possible pooling of types.

We can analogously define Negatively Assortative Matching (NAM) by requiring $(i_1 - i_2)(j_1 - j_2) \leq 0$. Under NAM, finding a higher a-side type in market 1 than in market 2 implies we must find a weakly lower b-side type there.

Importantly, separation and assortative matching are logically distinct properties. Consider a simple example with types $\mathbb{I}^s = \{0, 1\}$ and three markets in T:

- Market 1: a-side type 0 matches exclusively with b-side type 0
- Market 2: a-side type 1 matches exclusively with b-side type 1
- Market 3: a mixture of a-side types 0 and 1 both match with b-side type 0

This equilibrium exhibits PAM: whenever we find a higher a-side type in one market than another, we never find a strictly lower b-side type there. However, it is not separating because market 3 pools different a-side types. Conversely, markets could be separating but match high types with low types, violating PAM.

3 Motivating Examples

Before characterizing the equilibrium outcomes, we introduce four motivating examples.

Marriage Market. Suppose the agents are looking for partners for marriage. Here we interpret types as the attractiveness of agents, and their roles as men and women. The utility from forming a marriage depends on the private type of both the individual and their marriage partner. As a parametric example, we assume the utility function is:

$$u^s(i,j) = \left(i^{\frac{\theta-1}{\theta}} + j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}.$$

In this example, $\theta > 0$ is the substitution elasticity. This utility function satisfies the common ranking assumption. Additionally, it is supermodular and satisfies increasing willingness-to-pay, two additional assumptions which we introduce in Section 4.

Labor Market. We can also use the same structure to obtain a model of the labor market. Spence (1973) assumed that workers' human capital was unobservable, while their education choices were observable. Here we assume that both firm productivity and worker human capital are unobservable and only transfers (i.e. wages) and queues (unemployment and vacancy rates) can be used to signal type. More precisely, assume side-a firms hire side-b workers. Then $u^a(i,j)$ is the output produced when a firm with productivity i hires a worker with human capital j and $u^b(j,o)$ is the nonpecuniary amenity that the worker derives from the match.

Market for Expertise. Suppose side-a is the seller of an asset while side-b is the buyer. Sellers hold a single asset which may be of two qualities, S or B, with equal probability. Quality S assets are naturally retained by the seller, while quality B assets are more valuable if purchased by the buyer. For example, the asset may be a mortgage and the quality may represent the difficulty of servicing the mortgage. The seller (mortgage issuer) may be better able to handle a hard-to-service loan, while the buyer (investment bank) may have the deep pockets to hold other loans. If the seller ultimately holds a equality S asset, their payoff is 1, while it is -1 if they hold a quality B asset. The buyer's payoffs are the opposite: 1 if they purchase a quality B asset and -1 if they purchase a quality S asset.

While the seller does not know the quality of the asset, he knows how easy it is to observe the quality, a number i. And while the buyer also cannot directly observe the quality of the asset, he knows his expertise in assessing asset quality, a number j. If a type i seller matches with a type j buyer for an asset with quality S, they observe a normally distributed random variable with mean -1 and precision $\tau(i,j)$, an increasing function. If it is instead an asset with quality B, they observe a normally distributed random variable with mean 1 and the same precision $\tau(i,j)$. They then jointly decide whether to trade the asset and make any specified transfers.[4]

It is straightforward to show that there is a gain from trading the asset if and only if the signal is positive. This means that the pre-transfer expected payoff of both the buyer and the seller from this meeting is

$$u(i,j) = \frac{1}{2} - \Psi(1, \tau(i,j)),$$

where $\Psi(1,\tau)$ is the probability that a normally distributed random variable with mean 1 and precision τ is negative. Since $\Psi(1,\tau)$ is decreasing in the precision τ, these payoffs are increasing in both i and j and, under appropriate restrictions on τ, may be supermodular.

To understand the payoffs, proceed as follows: If the asset is of quality S, with probability $\frac{1}{2}$, trade occurs with probability $1 - \Psi(-1,\tau) = \Psi(1,\tau)$, resulting in payoff -1 for the buyer. If the asset if of quality B, also with probability $\frac{1}{2}$, trade occurs with probability $1 - \Psi(1,\tau)$, resulting in payoff 1 for the buyer. If there is no trade, the buyer's payoff is zero. Adding this up gives the buyer's expected payoff. We can find the seller's expected payoff symmetrically.

Communicable Disease. Suppose individuals are looking to interact, and their types are their probability of being healthy. If a healthy individual interacts with the sick individual, they may become sick, incurring an expected cost κ, the product of the probability of getting sick and the cost of being sick. Additionally, all individuals, healthy or sick individuals gain 1 from an interaction. Sick people do not care whether they interact with sick or healthy people. All individuals know the probability that they are healthy, their type.

[4] This example is inspired by Farboodi, Kondor and Kurlat (2025).

In summary, if a side-s agent who is healthy with probability i matches with an individual who is healthy with probability j, the value of the interaction is $u^s(i,j) = 1 - \kappa i(1-j)$. More specifically, the probability that the individual gets sick is proportional to the product of the probability that they are healthy i and that their partner is sick j. This payoff structure is similar to Philipson and Posner (1993) (for HIV/AIDS) and Farboodi, Jarosch and Shimer (2021) (for COVID-19). Interestingly, while the the payoff function is increasing in the partner's probability of being healthy and is supermodular, it is decreasing in the own probability of being healthy, since healthy people stand to lose more from interacting with sick people. Thus it satisfies decreasing willingness-to-pay, the flip side of increasing willingness-to-pay and another assumption we introduce in Section 4.

4 Characterization of Equilibrium

This section develops some preliminary theoretical results. We first prove that finding a partial equilibrium is equivalent to solving a particular optimization problem. We then show that market utilities must be continuous in a CSE and use that to establish that under supermodularity, all equilibria must be separating. Finally, we establish conditions under which market utilities are monotonic in agent types, allowing us to characterize equilibria through a system of differential equations.

4.1 The Platform's Problem

It is standard in the competitive search literature to reformulate equilibrium as an optimization problem. We prove here that our formulation is amenable to such a transformation, providing a tractable way to find equilibria and establish their properties. More precisely, we show that any terms-of-trade offered in equilibrium must maximize platform profits subject to constraints which ensure that the promised mix of agents come to the terms-of-trade.

Given market utilities U^s, we consider the following optimization problem:

$$\max_{\substack{(\lambda^a,\lambda^b)\in\mathbb{A},\\ \{\phi^s,G^s\}_{s=a,b}\in\mathbb{T}}} m(\lambda^a,\lambda^b)(\phi^a+\phi^b) \tag{7}$$

$$\text{s.t.}\quad U^s(i) \geq \lambda^s\left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j)dG^{\bar{s}}(j) - \phi^s\right)\ \forall i\in\mathbb{I}^s, s\in\{a,b\},$$

$$\int_{\mathbb{I}^s} U^s(i)dG^s(i) = \int_{\mathbb{I}^s}\lambda^s\left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j)dG^{\bar{s}}(j) - \phi^s\right)dG^s(i),\ s\in\{a,b\}.$$

The objective function is $V(\tau,\lambda^a,\lambda^b)$ defined in equation (2), the platform's expected gross profit per unit of advertising effort. The first constraint says that the market utility of a type-i, side-s agent bound their utility $\bar{U}^s(i,\tau,\lambda^s)$ above, where utility is defined in equation (1). The second constraint requires that agents who do participate (those in the support of G^s) receive exactly their market utility.

Our first result establishes the close link between a solution to this optimization problem and a partial equilibrium:

Lemma 1 *Given U^a, U^b, a partial equilibrium $\{T^p, T, (\Lambda^s, U^s)_{s=a,b}\}$ can be constructed as follows:*

- *T^p is the set of all $\tau = (\phi^s, G^s)_{s=a,b}$ for which there exist $(\lambda^s)_{s=a,b} \in \mathbb{\Lambda}$ satisfying the constraints of Problem (7);*
- *For $\tau \in T^p$, $\Lambda^s(\tau)$ is the corresponding λ^s from the constraints of (7);*
- *T is the set of $\tau \in T^p$ such that $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ solves Problem (7).*

Conversely, for any partial equilibrium $\{T^p, T, (\Lambda^s, U^s)_{s=a,b}\}$ and any $\tau = (\phi^s, G^s)_{s=a,b} \in T$, the tuple $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ solves problem (7).

The proof is in Appendix A.

While it is always useful to restructure a partial equilibrium as an optimization problem, problem (7) remains challenging to solve: it is a mathematical program with equilibrium constraints, nonconvex because increasing $dG^s(i)$ from zero changes inequality constraints into equality constraints. In the remainder of the paper, we characterize the outcome of the model by imposing additional structure on preferences and by using the notion of CSE, rather than just a partial equilibrium.

4.2 Supermodularity and Separation

To start, we prove that market utility is continuous in any CSE.

Lemma 2 *In any CSE, U^s is continuous for $s \in \{a, b\}$.*

The proof is in Appendix A. Continuity follows from agents' ability to mimic the behavior of nearby types. More precisely, since the payoff function u^s is continuous in its first argument, individuals can imitate the strategy of nearby types and get almost the same level of utility. In a CSE, all types must receive their market utility at some profit-maximizing terms-of-trade, and so that level of market utility inherits the continuity of the payoff function.

Next, we prove that if preferences are supermodular, all terms-of-trade used in a CSE only attract a single type of agent on each side of the market. First, we state the supermodularity assumption:

Assumption 2 (Supermodularity) *For every $s \in \{a,b\}$, $i, i' \in \mathbb{I}^s$ and $j, j' \in \mathbb{I}^{\bar{s}}$ with $i > i'$ and $j > j'$, $u^s(i,j) + u^s(i',j') > u^s(i,j') + u^s(i',j)$.*

Supermodularity implies that higher types have a stronger preference for matching with higher partner types. Proposition 1 below establishes that under Common Ranking and Supermodularity, any terms-of-trade used in a CSE solves the following optimization problem:

$$\begin{aligned} \max_{(\lambda^a,\lambda^b)\in\mathbb{A},\{\phi^s,k^s\}_{s=a,b}} \; & m(\lambda^a,\lambda^b)(\phi^a+\phi^b) \\ \text{s.t.} \;\; & U^s(i) \geq \lambda^s(u^s(i,k^{\bar{s}}) - \phi^s) \;\forall i < k^s, i \in \mathbb{I}^s, s \in \{a,b\} \\ & U^s(k^s) = \lambda^s(u^s(k^s,k^{\bar{s}}) - \phi^s) \;\forall s \in \{a,b\}, \\ & \phi^a + \phi^b \geq 0. \end{aligned} \tag{8}$$

We call the first constraint in problem (8) the *downward incentive constraint*, since it ensures that type i on side s does not want to come to the terms-of-trade intended for type $k^s > i$. The second constraint is the *participation constraint*, which ensures that type k^s earns their market utility at this terms-of-trade.

Problem (8) differs from the problem (7) in two important ways. First, it considers only separating terms-of-trade, in which exactly one type from each side participates in each market. Second, it includes only the downward incentive constraints. We ignore the upward incentive constraints, whcih ensure that high types do not deviate to terms-of-trade intended for lower types.

Proposition 1 *Assume Common Ranking and Supermodularity. A tuple $\{T^p, T, (\Lambda^s, U^s)_{s=a,b}\}$ together with a measure μ on T constitutes a CSE if and only if:*

1. *For each $\tau = (\phi^a, \phi^b, G^a, G^b) \in T$, there exist types $(k^a, k^b) \in \mathbb{I}^a \times \mathbb{I}^b$ such that the associated distributions are degenerate: $dG^a(k^a) = dG^b(k^b) = 1$;*

2. *For each $\tau = (\phi^a, \phi^b, G^a, G^b) \in T$, the fees ϕ^s and matching probabilities $\lambda^s = \Lambda^s(\tau)$ solve problem (8) given the types (k^a, k^b) and market utilities U^s, with maximized value equal to c;*

3. *The set of possible terms-of-trade T^p and matching probabilities Λ^a, Λ^b satisfy the optimal search and impossible terms-of-trade conditions from Definition 1;*

4. *The measure μ satisfies the market clearing condition and U^s satisfies the market utility condition from Definition 2.*

The proof is in Appendix A.

The first two parts of Proposition 1 characterize the profit-maximizing terms-of-trade in a CSE under Supermodularity. First, such terms-of-trade attract one type from each side of the market, rather than a mixture over types. Second, such terms-of-trade solve problem (8), which in particular includes the constraints for keeping out lower types but drops the constraints from keeping out higher types. The last two parts of the Proposition tell us how to find the other elements of a CSE.

This Proposition yields two important implications for equilibrium structure. First, they immediately establish our main separation result:

Corollary 1 *Assume Common Ranking and Supermodularity. Any CSE is separating.*

To understand why pooling cannot occur in equilibrium, consider a market where multiple types participate on at least one side. Common Ranking implies that all participants would prefer to match with only the highest types from the other side of the market. This suggests creating a new market charging higher fees to attract only these high types. Supermodularity guarantees that lower types would be unwilling to pay the fees that make higher types indifferent, because the higher types value the improved matching enough to pay more than lower types could afford. While such a market might attract even higher types than those in the original pooling market, potentially making it infeasible, this just creates further profit opportunities. Even higher fees could be charged to attract those higher types. This process continues until complete separation is achieved.

An example illustrates the need for Supermodularity. Consider a market with $\mathbb{I}^s = \{0, 1\}$ and payoffs $u^s(0, 0) = 1$, $u^s(0, 1) = 1.5$, $u^s(1, 0) = 1.1$, and $u^s(1, 1) = 1.2$ for $s \in \{a, b\}$. These payoffs satisfy Common Ranking but not Supermodularity. With matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ on the unit simplex ($\lambda^a \geq 0$, $\lambda^b \geq 0$, and $\lambda^a + \lambda^b \leq 1$), market utilities $U^s(0) = 0.33$ and $U^s(1) = 0.25$, and appropriate population distributions ($F^s(0) = 0.124$ and $c \approx 0.293$), we find a CSE with pooling. Specifically, a single market attracts both types on both sides, with $G^s(0) = F^s(0)$, $\lambda^s \approx 0.320$, and $\phi^s \approx 0.406$ for $s \in \{a, b\}$.

Second, the Proposition establishes that only the constraint of excluding lower types can bind in equilibrium. More precisely, we say *all upward incentive constraints are slack* in a CSE if for any active market $\tau = (\phi^s, G^s)_{s=a,b} \in T$ and type i in the support of G^s, $U^s(i') > \bar{U}^s(i', \tau, \Lambda^s(\tau))$ for all $i' > i$.

Corollary 2 *Assume Common Ranking and Supermodularity. In any CSE, all upward incentive constraints are slack.*

Slack upward incentive constraints are especially useful for characterizing equilibria with positive assortative matching (PAM), as it allows us to solve for market utilities recursively starting from the lowest type.

This result contrasts with the counterveiling incentives literature (Lewis and Sappington, 1989), which emphasizes how heterogeneity in individuals' outside option may lead to incentive constraints that bind in either direction. Here the outside option is individuals' market utility. In partial equilibrium, where market utility is exogenous, upward incentive constraints may bind here as well. But in a CSE, competition between platforms determines market utility. Under Common Ranking and Supermodularity, this competition ensures that only lower types are tempted to come to the market intended for higher types.

4.3 Monotonicity of Market Utility

To this point, we have assumed imposed restrictions on how $u^s(i,j)$ varies with j (it is increasing under Common Ranking) and how the derivative of $u^s(i,j)$ with respect to i varies with j (it is increasing under Supermodularity). We have not yet made any assumptions about how $u^s(i,j)$ varies with i. In the remainder of the paper, we find it useful to distinguish between two cases, with the qualitative characterization of equilibrium depending in important ways on which case we are in.

First, we consider the possibility that $u^s(i,j)$ is increasing in i for all j:

Assumption 3 (Increasing Willingness-to-Pay (IWTP)) $\forall s = a, b, i > i' \in \mathbb{I}^s, j \in \mathbb{I}^{\bar{s}}$, $u^s(i,j) \geq u^s(i',j)$.

We call this Increasing WTP because it implies that higher types value matching with any given partner j more than lower types do. The opposite assumption is that $u^s(i,j)$ is decreasing in i for all j:

Assumption 4 (Decreasing Willingness-to-Pay (DWTP)) $\forall\, s = a, b,\ i > i' \in \mathbb{I}^s,\ j \in \mathbb{I}^{\bar{s}},\ u^s(i,j) \leq u^s(i',j)$.

This is decreasing WTP because higher types value matching with any given partner j less than lower types do. We refer to these assumptions collectively as monotone WTP. Monotone WTP is a standard assumption, but conceptually distinct from Common Ranking and Supermodularity.

Under Monotone WTP, market utility inherits the monotonicity of underlying preferences:

Lemma 3 *Assume Common Ranking. In any CSE:*

1. *if there is IWTP, then $U^s(i)$ is weakly increasing in i.*

2. *If there is DWTP, then $U^s(i)$ is weakly decreasing in i.*

The proof is again in Appendix A. The proof essentially works by eliminating the fees from the constraints in problem (7).

$$U^s(i') - U^s(i) \geq \lambda^s \int_{\mathbb{I}^{\bar{s}}} (u^s(i', j) - u^s(i, j)) dG^{\bar{s}}(j)$$

for any terms-of-trade that attracts type-i, side-s agents and any other i'. With IWTP, the right hand side is positive for all $i' > i$, while with DWTP it is positive for all $i' < i$, proving the result. This monotonicity result provides the foundation for our subsequent analysis of equilibrium matching patterns, where we turn to continuous types.

4.4 Continuous Type Distribution and First-order Approach

We have established that all equilibrium markets must be separating under Common Ranking and Supermodularity (Proposition 1). We have also proved that market utility U^s is monotone under Monotone WTP (Lemma 3). We now develop tools for characterizing these markets when types are continuously distributed. We focus on the case where the support of the type distribution I^s is a nonempty interval $[\underline{i}, \bar{i}]$.

Our first step is to establish sufficient regularity of market utilities to justify a local approach to incentive constraints:

Lemma 4 *Assume Common Ranking, Supermodularity, and Monotone Willingness-to-Pay. If $\mathbb{I}^s = [\underline{i}, \bar{i}]$ for $s = a, b$, then in any CSE, market utility $U^s(i)$ is almost everywhere differentiable. For any active market $\tau \in T$ that attracts types (k^a, k^b), if $k^s \in (\underline{i}, \bar{i})$ and U^s differentiable at k^s,*

$$U^{s\prime}(k^s) = \Lambda^s(\tau) u_1^s(k^s, k^{\bar{s}}),$$

where u_1^s denotes the partial derivative of u^s with respect to its first argument.

The proof is in Appendix A.

For any market attracting types k^a and k^b with U^s differentiable at k^s, we can use this lemma to replace the downward incentive constraints in Problem (8) with the *local incentive constraint*:

$$\max_{(\lambda^a, \lambda^b) \in \mathbb{A}, \{\phi^s, k^s\}_{s=a,b}} m(\lambda^a, \lambda^b)(\phi^a + \phi^b) \tag{9}$$

$$\text{s.t. } U^{s\prime}(k^s) = \lambda^s u_1^s(k^s, k^{\bar{s}}) \ \forall s \in \{a, b\}$$

$$U^s(k^s) = \lambda^s (u^s(k^s, k^{\bar{s}}) - \phi^s) \ \forall s \in \{a, b\},$$

$$\phi^a + \phi^b \geq 0.$$

We can then eliminate the fees ϕ^a, ϕ^b from this problem using the participation constraint and the contact rates λ^a, λ^b using the local incentive constraint. This gives us the gross value of advertising terms-of-trade to attract (k^a, k^b):

$$\hat{V}(k^a, k^b) = m\left(\frac{U^{a\prime}(k^a)}{u_1^a(k^a, k^b)}, \frac{U^{b\prime}(k^b)}{u_1^b(k^b, k^a)}\right) \sum_{s=a,b} \left(u^s(k^s, k^{\bar{s}}) - \frac{U^s(k^s)u_1^s(k^s, k^{\bar{s}})}{U^{s\prime}(k^s)}\right). \tag{10}$$

In order for a platform advertising this terms-of-trade to obtain this value, it must satisfy three conditions. First, the combination of matching probabilities must be feasible:

$$\left(\frac{U^{a\prime}(k^a)}{u_1^a(k^a, k^b)}, \frac{U^{b\prime}(k^b)}{u_1^b(k^b, k^a)}\right) \in \mathbb{\Lambda}.$$

Second, the sum of the fees must be nonnegative:

$$\sum_{s=a,b} \left(u^s(k^s, k^{\bar{s}}) - \frac{U^s(k^s)u_1^s(k^s, k^{\bar{s}})}{U^{s\prime}(k^s)}\right) \geq 0.$$

And finally all of the downward incentive constraints must be satisfied, not just the local incentive constraint. If any of these conditions fails, there can not be any terms-of-trade which matches k^a and k^b at this level of market utility.

Equation (10) captures how local incentive constraints shape the platform's profits. The matching rates are determined by the elasticity of agents' market utility U^s relative to their payoff u^s, while the total surplus must cover both the direct utility from matching and an information rent term reflecting agents' local incentive constraints. This expression reveals important features of the equilibrium construction. When $k^a, k^b > 0$ and market utilities $U^s(i)$ are known, $\hat{V}(k^a, k^b)$ is completely determined by incentive constraints. There is no optimization problem left to solve. The local incentive constraints pin down the matching rates, and these in turn determine the platform's revenue through the matching probability m. Thus, finding active markets reduces to identifying pairs (k^a, k^b) that yield the highest value of $\hat{V}(k^a, k^b)$, subject to market clearing.

However, some challenges remain. First, we need to determine the market utilities $U^s(i)$ themselves. Second, we need to verify that all downward incentive constraints are satisfied, not just local ones. Third, markets that attract the lowest types ($k^a = 0$ or $k^b = 0$) require special treatment because there are no downward incentive constraints and so local incentive constraints don't bind for these types. These markets must solve the unconstrained platform problem, which then provides boundary conditions for the market utilities of higher types. Similarly, we can only compute $\hat{V}(k^a, k^b)$ at points where U^a and U^b are differentiable, which we have so far established is true almost everywhere.

In the next two sections, we tackle these two challenges, studying first Increasing WTP (Section 5) and then Decreasing WTP (Section 6). In both sections, we assume advertising costs are zero, $c = 0$. This gives us a sharp characterization of necessary and sufficient conditions for a PAM CSE, as well as a sharp characterization of when matching probabilities or matching fees are used to separate types in equilibrium. In Section 7, we show how our results carry over to an environment with positive advertising costs.

5 IWTP with Zero Advertising Cost

Throughout this section, we consider an environment satisfying Common Ranking, Supermodularity, and IWTP. We also assume advertising costs are zero, $c = 0$. Finally, we focus on a symmetric environment, which simplifies our exposition:

Assumption 5 (Symmetric Environment) *An environment has symmetry if*

1. *equal populations on each side:* $I^a = I^b = I$*;*
2. *common type distributions:* $F^a = F^b = F$ *with support* $\mathbb{I}^a = \mathbb{I}^b = \mathbb{I}$*;*
3. *symmetric match utilities:* $u^a(i,j) = u^b(j,i) = u(i,j)$ *for all* i, j*; and*
4. *symmetric matching technology:* $m(\lambda^a, \lambda^b) = m(\lambda^b, \lambda^a)$.

We then look for a symmetric CSE with PAM, meaning it satisfies three properties:

Definition 5 (Symmetric PAM Competitive Search Equilibrium) *A symmetric PAM Competitive Search Equilibrium is a competitive search equilibrium satisfying:*

1. *type-i, side-a agents match with type-i, side-b agents for all* $i \in \mathbb{I}$*;*
2. *same-type agents on different sides have the same market utility:* $U^a(i) = U^b(i) = U(i)$ *for all* $i \in \mathbb{I}$*;*
3. *markets for same-type agents have symmetric terms: In the active market* τ *that attracts type-i agents on both sides,* $\lambda^a(\tau) = \lambda^b(\tau) \equiv \ell(i)$ *and fees* $\phi^a(\tau) = \phi^b(\tau) \equiv \Phi(i)$.

5.1 Leading Example

We begin with a parametric example, which illustrates our general results. Assume that the utility function has a constant elasticity of substitution,

$$u^s(i,j) = \left(\frac{1}{2} i^{\frac{\theta-1}{\theta}} + \frac{1}{2} j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

for $s = a, b$ and some $\theta > 0$. When $\theta = 1$, this reduces to the multiplicatively separable case, $u^s(i,j) = \sqrt{ij}$. As $\theta \to 0$, the payoff function is Leontief, $u^s(i,j) = \min\{i,j\}$, while as $\theta \to \infty$, the payoff function is $u^s(i,j) = \frac{i+j}{2}$. The type distribution is still uniform on $[\underline{i}, \bar{i}]$, with a measure 1 of agents on each side of the market.

We also assume the matching function has a constant elasticity of substitution,

$$m(\lambda^a, \lambda^b) = (1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma)^{\frac{1}{\gamma}},$$

where $\gamma > 0$ and $1/(1+\gamma)$ is the elasticity of substitution in the matching function $M(n^a, n^b)$. The support of the possible agent matching rates is $\mathbb{\Lambda} = \left\{(\lambda^a, \lambda^b) | 1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma \geq 0\right\}$.

This is a Symmetric Environment. Proposition 2 below implies that there is a symmetric PAM CSE if and only if $\theta \leq 1 + \gamma$ and the lower bound on the type distribution is zero, $\underline{i} = 0$. The first condition is the same as one would obtain from the complete information setting (Eeckhout and Kircher, 2010): PAM is sustained when the elasticity of substitution in either the matching function or production function is small.[5][6] The second condition, $\underline{i} = 0$, is novel to an environment with private information. We next show what happens both when $\theta \leq 1 + \gamma$ and $\underline{i} = 0$, and then when one of these conditions fails.

Positive sorting When $\theta \leq 1 + \gamma$ and $\underline{i} = 0$, there is a symmetric PAM CSE. In this competitive search equilibrium, market utility is $U(i) = \bar{\lambda} i/2$, where $\bar{\lambda} \equiv 2^{-1/\gamma}$ and hence satisfies $m(\bar{\lambda}, \bar{\lambda}) = 0$. Additionally, a terms-of-trade τ is active if and only if it matches type i agents to type i agents, charging a fee of $\frac{i}{2}$. In all active markets, agents' matching probability is $\bar{\lambda}$. For a proof of these claims, see Proposition 2.

In words, platforms send out enough (free) advertising so as to saturate the market, driving the platform matching probability to zero. However, agents must still pay a fee when they match. These fees consume half of match value. They exist only because of private information and are dissipated across the platforms by the saturation of advertising,

[5] We formally introduce the complete information model in the Appendix G.

[6] For the exact inequality in the complete information setting, see equation (15) in Eeckhout and Kircher (2010). It is straightforward to prove that this is equivalent to $\theta \leq 1 + \gamma$ in our setting, in Appendix G.

driving platform profits to zero.

Figure 1 illustrates the CSE both with private information (red solid lines) and with observable types (blue dashed lines) when $\theta = 1$ and $\gamma = 1$. The top left panel shows the partner for each type: each agent matches with someone of the same type on the other side of the market, both with private information and with observable types. The top right panel shows market utility: private information cuts it in half. The middle row shows the matching probability for agents on the left (0.5 in both environments) and for platforms on the right (0 in both environments). The bottom row shows the fees. With private information, each agent pays $i/2$ (left) and so the platform collects i (right). With observable type, competition drives the fees to zero.

Thus in this case, the market uses fees alone to separate types. Higher types pay higher fees, set a level to ensure that lower types do not want to come to the market. The market does not ration using the matching probability at all. The reason for this is intuitive: with IWTP, higher types value matching more, and so are willing to pay a premium to match, i.e. a high fee. But since high types value matching more, they are less willing to be rationed. This means that rationing is only a useful way of signaling that one has a low type. Since incentive constraints can only bind downward (Corollary 1), fees rather than rationing prevail in equilibrium.

Cross-Subsidized Matching When $\theta < 1 + \gamma$ and $\underline{i} > 0$, the PAM CSE breaks down. To understand why, it helps to start from a conjectured equilibrium analogous to the one described above: suppose market utility is $U(i) = \bar{\lambda}(i + \underline{i})/2$, i always matches with i, fees in the (i, i) terms-of-trade satisfy $\Phi(i) = (i - \underline{i})/2$, and agent matching probabilities are at the maximum feasible value $\bar{\lambda}$ in all active markets. As in the case where $\underline{i} = 0$, there is no distortion for the lowest type $\underline{i}$, but otherwise local incentive constraints are binding.

But this is not a competitive search equilibrium. At this level of market utility, it is always profitable for a platform to offer a terms-of-trade matching $\underline{i}$ to some $j > \underline{i}$.[7] A $(\underline{i}, j)$ market wasn't profitable when the lowest type was zero because $\underline{i}$ faced downward incentive constraints. But when the lowest type is $\underline{i}$, those constraints are gone and such terms-of-trade are profitable.

For a competitive search equilibrium to exist, the competition between platforms must some how boost $U(\underline{i})$ above $\bar{\lambda}\underline{i}$. Since there are no information frictions in the $(\underline{i}, \underline{i})$, it is not feasible to do this without changing who $\underline{i}$ matches with. The argument in the previous paragraph suggests how competitive platforms do this: by matching $\underline{i}$ with some type $j > \underline{i}$.

Figure 2 illustrates the competitive search equilibrium when $\theta = \gamma = 1$ and types are

[7] Formally, $\partial \hat{V}(\underline{i}, j)/\partial j|_{j=\underline{i}} > 0$ at this level of market utility.

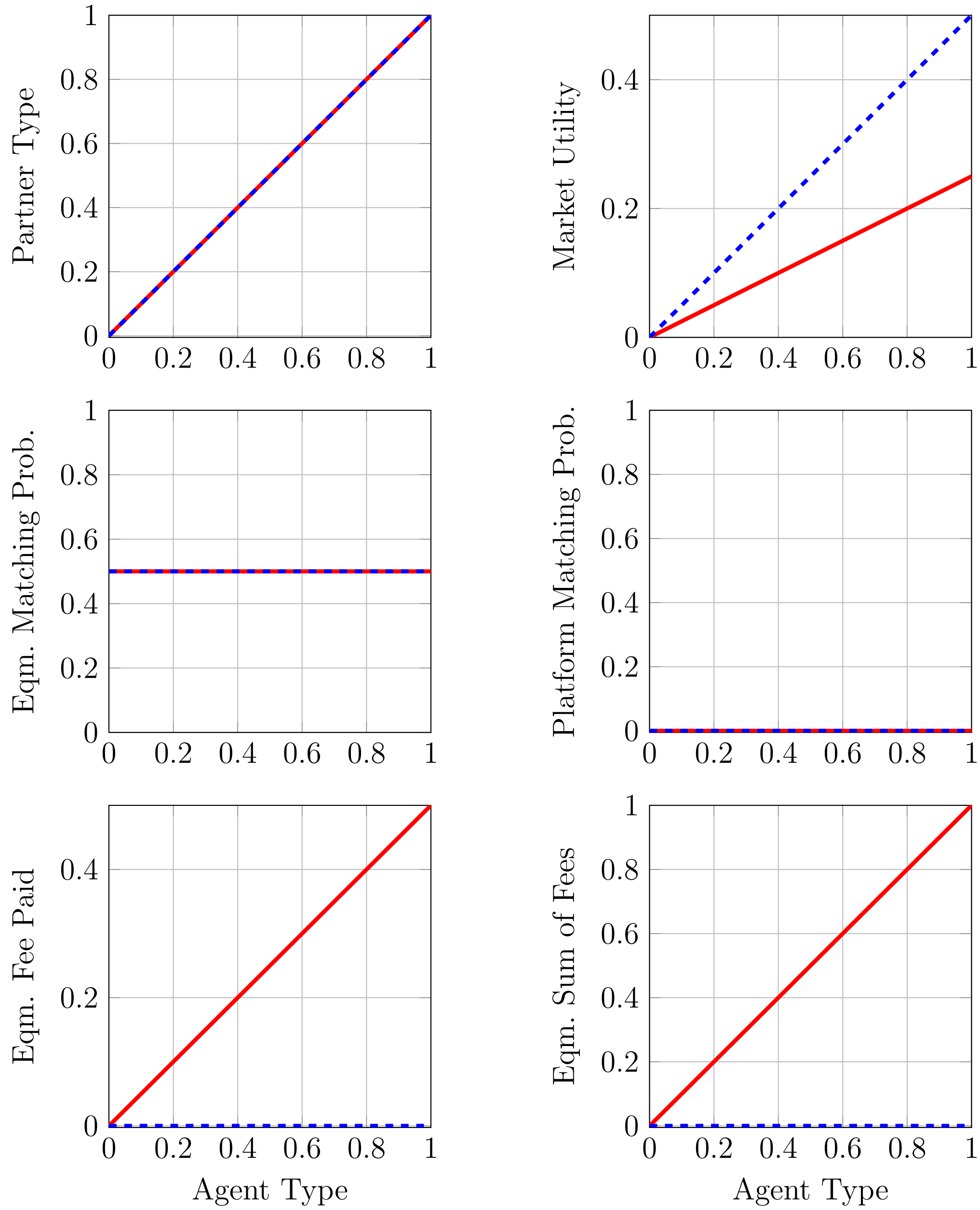

Figure 1: Positively Sorted Equilibrium. Notes: Red solid lines represent private-information equilibrium outcomes, while blue dashed lines show observable-type equilibrium outcomes. Parameters: $\theta = 1$, $\gamma = 1$, types are distributed uniformly on $[0, 1]$.

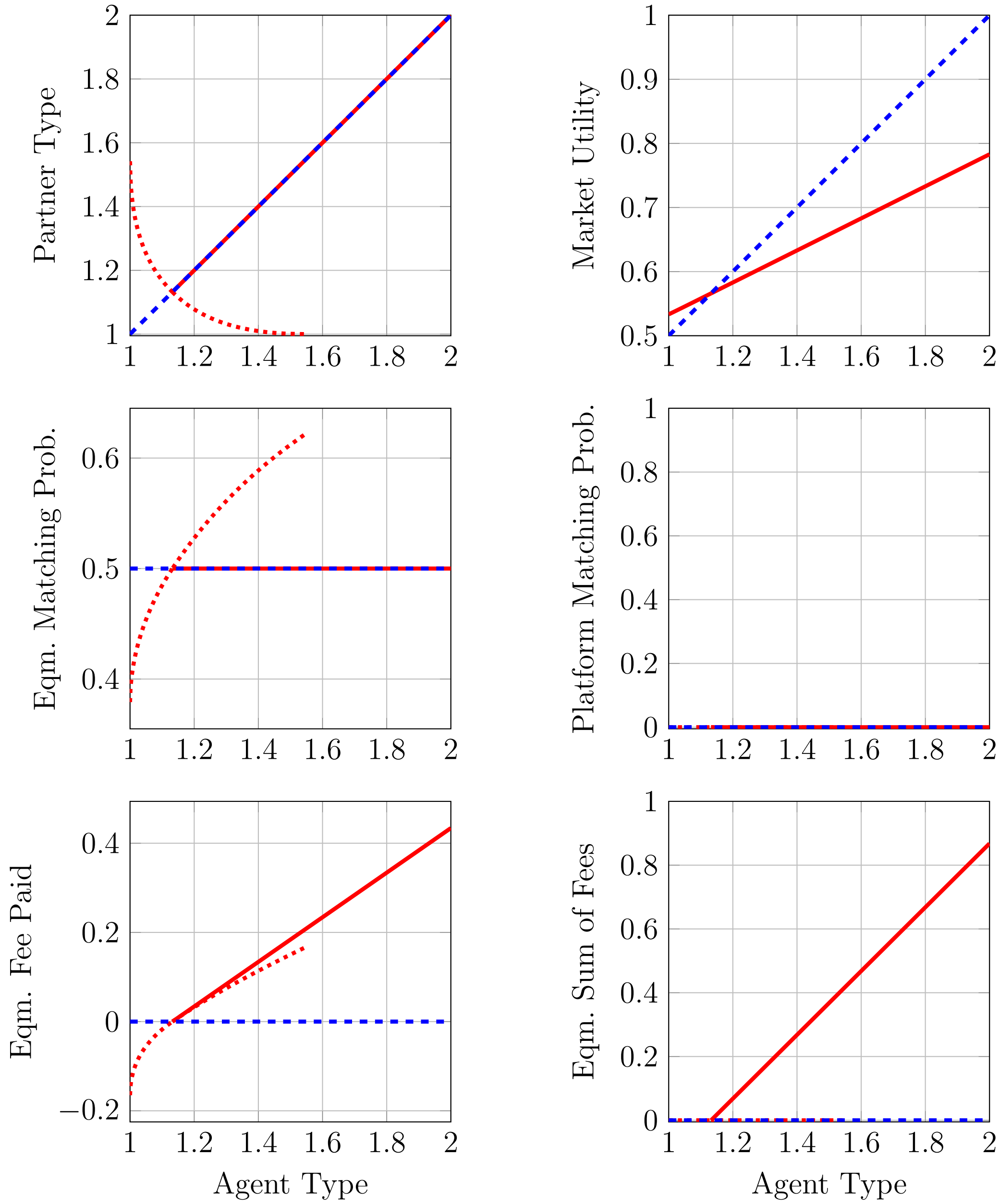

Figure 2: Cross-Subsidized Equilibrium. Notes: Red solid and dotted lines represent private-information equilibrium outcomes, while blue dashed lines show observable-type equilibrium outcomes. Parameters: $\theta = 1$, $\gamma = 1$, types are uniform on $[1, 2]$.

uniform on $[1, 2]$. Type 1 agents on one side of the market match with type 1.54 agents on the other side of the market. In these meetings, type 1 agents match with a low probability, 0.38, while type 1.54 agents match with complementary probability 0.62 and the platform matching probability is zero. These differences in matching probabilities are not a choice, but instead are dictated by the local incentive constraints (Lemma 4): Since u is supermodular, $u_1(1.54, 1) < u_1(1.54, 1.54)$, raising 1.54's matching probability when matched with 1. Additionally, the high type pays for the higher matching probability, making a transfer to the low type via the platform: the high type pays a fee 0.17 to the platform, while type 1 receives a payment of the same magnitude. Altogether, these transfers raise the market utility of type 1 from 0.5 (the highest they can achieve when matched to other type 1s) up to 0.53.

But that is not the end of the story. Once type 1 has higher market utility, the incentive constraint is relaxed for slightly higher types, making matching with them more profitable. Equilibrium requires a further reassignment of matching partners. For example, type 1.01 is on one side of a platform, with 1.40 on the other side. The higher type matches with a higher probability and pays a positive fee, while the lower type matches with the complementary probability and is paid a fee of equal magnitude. The platform matches with zero probability and the sum of fees is zero. This pattern continues until we reach type 1.13, which matches with the same type with probability 0.5 and pays no fees. The red dotted lines in Figure 2 trace out these matches.

With a uniform type distribution, these negatively sorted matches absorb all the agents between 1 and 1.13 but only some of the agents between 1.13 and 1.54. The remaining agents between 1.13 and 1.54, as well as all higher types of agents, engage in positive assortative matching, exactly as in the case with $\underline{i} = 0$. At these terms of trade, every agent matches with probability 0.5 and pays half of the difference $i - 1.13$ in fees. We prove in Appendix B that this is a competitive search equilibrium. In particular, all active markets satisfy the relevant incentive constraints and it is unprofitable to create any new market.

Competing platforms have effectively cross-subsidized all agents $i \in [1, 1.13)$ by matching them with a higher type and charging them a negative fee. This has the effect of boosting the market utility of such types above what they would achieve in the economy without private information. That is, these lower types earn an information rent paid for through competitive cross-subsidization. Once these types are subsidized, the incentive constraints on higher types is relaxed, boosting their market utility above the benchmark of $\bar{\lambda}(i + \underline{i})/2$ described above.

To our knowledge, the insights that competing platforms can cross-subsidize agents in a separating equilibrium is novel. We stress that there is no platform that thinks of itself as cross-subsidizing some type of agent. This is impossible in a CSE, because some other platform would cream-skim the agents that are paying for the cross-subsidization. Instead, the competition across platforms creates this cross-subsidization by changing who matches with whom, highlighting the interplay between private information and sorting.

The equilibrium features a second novelty: all agents in the interval $(1.13, 1.54]$ are indifferent about matching both with an agents of their own type and with a much lower type of agent, someone with $i \in [1, 1.13)$. That is, there is an open interval of types that must split their search between two distinct markets in a CSE.

Negative sorting Finally, we look at what happens when $\theta > 1+\gamma$. In this case, regardless of the value of $\underline{i}$, we find that there is negative assortative matching (NAM) with both private information and observable types. Figure 3 depicts the equilibrium. We show how to characterize it in Appendix E.2.

Many patterns are similar with both private information and observable types. There is negatively assortative matching, where higher types match systematically with lower types, though the exact matching pattern differs between the two environments. Higher types also match with a higher probability, while lower types pay a higher fee. Notably, the platform matching probability is always zero, while the sum of fees collected is zero with observable types but strictly positive when there is private information (except in the $(0, 1)$ market).

5.2 Generalization of Results

The examples we just went through suggest two general takeaways: (1) in a CSE, platforms collect fees to screen agents, but their matching probability is always zero; and (2) positive sorting requires both sufficient complementarity in the production technology and some special condition to handle the slack information constraint for the lowest type. Here we prove that these results hold more generally. We start with the platform matching probability.

Lemma 5 *Assume Common Ranking, Supermodularity, and IWTP, as well as $c = 0$. For any profit-maximizing terms-of-trade $\tau \in T$ in a CSE, $m(\Lambda^a(\tau), \Lambda^b(\tau)) = 0$.*

The proof is in Appendix B. Intuitively, in a competitive search equilibrium, platforms must break even. If a platform has a positive matching probability, that means the sum of fees must be zero. So consider a different platform that tries to attract the same types at slightly higher fees. To keep attracting that type, the higher fees must be offset by a higher agent matching probability and hence a lower platform matching probability, but this is possible since the platform had a positive matching probability. But of course this may not be

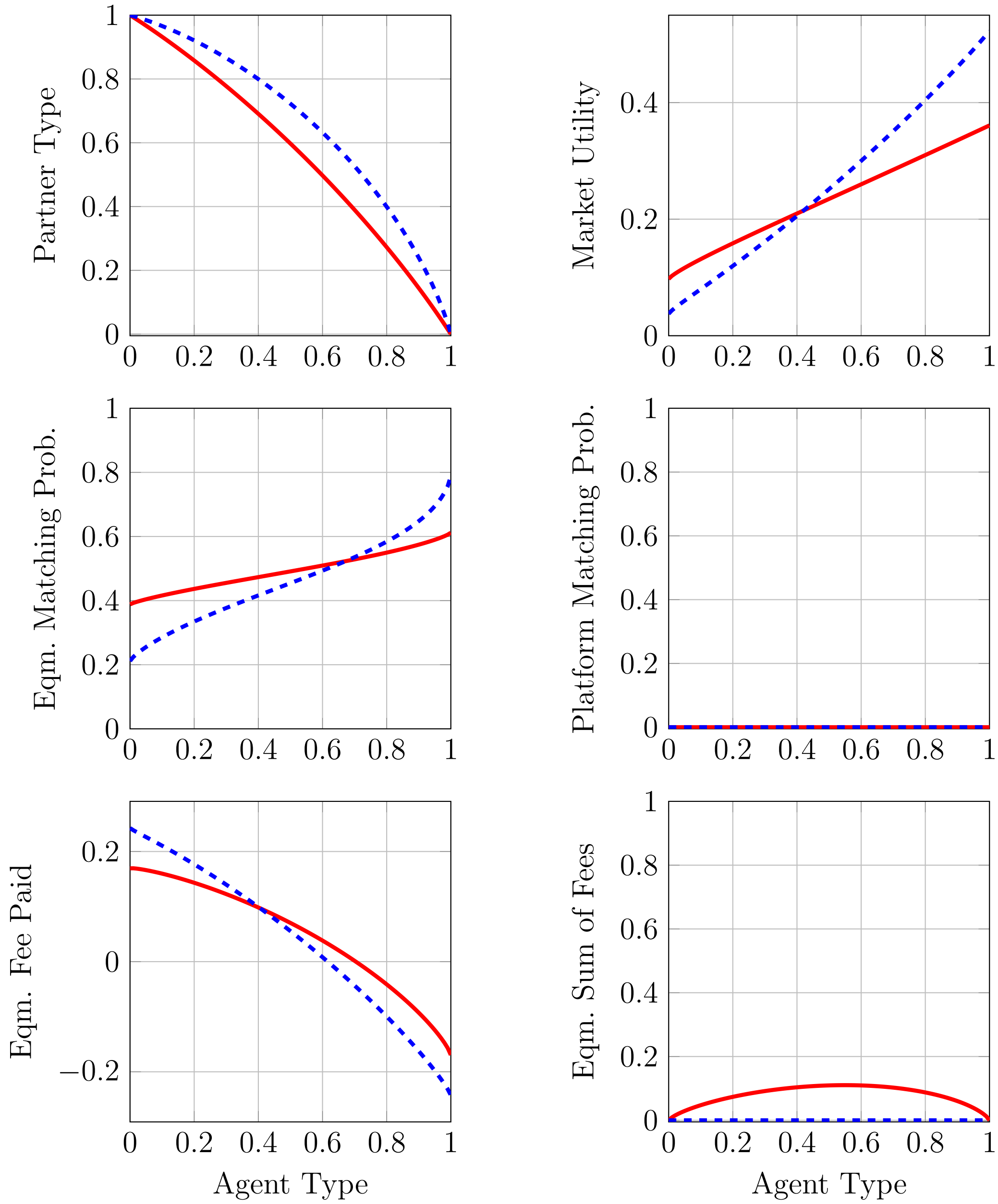

Figure 3: Negatively Sorted Equilibrium. Notes: Red solid lines represent private-information equilibrium outcomes, while blue dashed lines show observable-type equilibrium outcomes. Parameters: $\theta = 5$, $\gamma = 1$, types are distributed uniformly on $[0, 1]$.

consistent with equilibrium: the change in fees may attract different types. With IWTP, it can only attract higher types, which is always beneficial under Common Ranking and Supermodularity. In this way, the we can find a new terms of trade that has higher fees and a positive matching probability, inconsistent with a CSE.

Next, we find conditions for a symmetric PAM CSE to exist:

Assumption 6 *Define $\bar{\lambda}$ such that $m(\bar{\lambda}, \bar{\lambda}) = 0$.*

(a) For all $i > j > \underline{i}$, $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}\right) \notin \mathbb{A}^o$;

(b) For all $i > \underline{i}$, either $u_1(i, \underline{i}) = 0$ or $u(\underline{i}, i) + \tilde{\phi}(i) \leq 0$ or $\left(\bar{\lambda}\frac{u(\underline{i},i)}{u(\underline{i},i)+\tilde{\phi}(i)}, \tilde{\ell}(i)\right) \notin \mathbb{A}^o$, where

$$\tilde{\phi}(i) \equiv u(i, \underline{i}) - \frac{u_1(i, \underline{i})}{u_1(i, i)}\left(u(\underline{i}, \underline{i}) + \int_{\underline{i}}^{i} u_1(k, k)dk\right) \quad \text{and} \quad \tilde{\ell}(i) \equiv \bar{\lambda}\frac{u_1(i, i)}{u_1(i, \underline{i})}.$$

Assumption 6(a) ensures that creating a market matching i and j with $i > j > \underline{i}$ is not profitable. In our CES example, the condition reduces to

$$\frac{1}{2}i^{-\frac{\gamma}{\theta}} + \frac{1}{2}j^{-\frac{\gamma}{\theta}} > \left(\frac{1}{2}\left(i^{-\frac{\gamma}{\theta}}\right)^{\frac{\theta-1}{\gamma}} + \frac{1}{2}\left(j^{-\frac{\gamma}{\theta}}\right)^{\frac{\theta-1}{\gamma}}\right)^{\frac{\gamma}{\theta-1}}$$

Using Jensen's inequality, one can verify that this holds if and only if $\theta < 1 + \gamma$.

Assumption 6(b) ensures that a terms-of-trade matching $\underline{i}$ with $i > \underline{i}$ is not profitable. Type $\underline{i}$ is different than the other types because they do not face any downward incentive constraints. Again, in our parametric example one can verify that Assumption 6(b) holds if and only if $\theta \leq 1 + \gamma$ and $\underline{i} = 0$.[8]

We now state our general characterization of PAM with IWTP:

Proposition 2 *Assume Common Ranking, Supermodularity, IWTP, Symmetric Environment and Assumption 6. If $c = 0$, there is a symmetric PAM CSE. For the terms of trade $\tau(i)$ attracting type i agents, the matching probability is $\Lambda(\tau(i)) = \bar{\lambda}$; the fee solves $\Phi'(i) = u_2(i, i)$ with $\Phi(0) = 0$; and market utility solves $U'(i) = \bar{\lambda}u_1(i, i)$ with $U(0) = \bar{\lambda}u(0, 0)$.*

The proof is in Appendix B. It proceeds as follows. First, we conjecture that there is PAM with $U'(i) = \bar{\lambda}u_1(i, i)$ with $U(0) = \bar{\lambda}u(0, 0)$. We find the the contact rates and fees in all (i, i) markets by solving a symmetric version of problem (9). This gives us that the matching

[8] Assume $\underline{i} = 0$. $\theta \leq 1$ implies $u_1(i, 0) = 0$ for all $i > 0$, so 6(b) is satisfied. For $\theta > 1$, $u_1(i, 0) = 2^{\theta/(1-\theta)} > 0$ and $u(0, i) = 2^{\theta/(1-\theta)}i$, both positive, while $\tilde{\phi}(i) = 0$. It follows that 6(b) holds if and only if $\left(0, \bar{\lambda}\frac{u_1(i,i)}{u_1(i,0)}\right) \notin \mathbb{A}^o$, or equivalently if $\theta \leq 1 + \gamma$. If $\underline{i} > 0$, all the conditions in Assumption 6(b) are always violated.

probabilities are $\bar{\lambda}$ while fees satisfy $\Phi'(i) = u_2(i,i)$ with $\Phi(\underline{i}) = 0$. Second, we verify that all the global incentive constraints are satisfied. This always true under Common Ranking, Supermodularity, and IWTP.

Finally, we prove that there is no profitable deviation through the creation of a new terms-of-trade. This step of the proof relies on Assumption 6. From Proposition 1, we know that we can look exclusively at terms-of-trade that attract a single type on each side. Assumption 6(a) rules out the profitability of a terms-of-trade that attracts $i > 0$ on one side and $j > 0$ on the other. At such terms of trade, the local incentive constraints imply $(\lambda^a, \lambda^b) \notin \Lambda^o$, which in turn means that the deviation cannot be profitable. If this condition fails, we may get negative assortative matching, as in Figure 3. A terms-of-trade matching $i > \underline{i}$ to $\underline{i}$ is different, however, because the local incentive constraint may be slack for the lowest type. We use Assumption 6(b) to rule out the profitability of such terms-of-trade. If this condition fails, we may get competitive cross-subsidization, as in Figure 2.

Conditional on having a PAM equilibrium, the equilibrium outcomes resemble the ones from the parametric case of two CESs. All types have the same matching probability $\bar{\lambda}$, which is the maximal symmetric matching probability feasible under the matching function. Since higher types gain more from matching with IWTP, trying to exclude lower types by rationing matches is counterproductive. On the other hand, fees are increasing to ensure that lower types, who gain less from matching with a higher type, are willing to match with their own type rather than paying a higher fee to match with a higher type.

We close this section by looking at the expertise model from Section 3. Here we assume

$$u(i,j) = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{ij/2}} e^{-t^2} dt,$$

so the variance of the signal from a type i seller and type b buyer is $1/(ij)$. This always satisfies Common Ranking and IWTP. It satisfies Supermodularity if $ij < 1$, and so we restrict the type space to the interval $[0, 1]$.

We find that Assumption 6(a) is satisfied only on a tighter domain, $[0, \bar{i}]$. For example, with a CES matching function with parameter γ, we require $\bar{i} \leq \left(\frac{1+\gamma-\sqrt{1+2\gamma}}{\gamma}\right)^{1/2} < 1$. Additionally, Assumption 6(b) is always satisfied because $u_1(i, 0) = 0$. Thus there is PAM whenver the support of the type distribution is $[0, \bar{i}]$. In this case, the matching probability is $\bar{\lambda} = 2^{-1/\gamma}$ for everyone, fees cost half of equilibrium output, $\Phi(i) = u(i,i)/2$, and half of market utility is lost to private information, $U(i) = \bar{\lambda} u(i,i)/2$. Conversely, when Assumption 6(a) is violated, there is no equilibrium with PAM.

6 DWTP with Zero Advertising Cost

We now turn to a Symmetric Environment satisfying Common Ranking, Supermodularity, and DWTP. We also assume advertising costs are zero, $c = 0$. We are interested in finding when a Symmetric PAM CSE exists, and characterizing a CSE more generally.

6.1 Leading Example

We again start with a parametric example. Let $u^s(i, j)$ satisfy

$$u^s(i, j) = 1 - \kappa i(1 - j)$$

for $s = a, b$, with support on some interval $\mathbb{I} \subseteq [0, 1]$ and $0 < \kappa < 4$.[9] This is our model of disease transmission. A type-i individual is healthy with probability i. They gain 1 from a match but suffer a utility loss if they get sick, which may happen whenever they match with a sick partner. κ represents the product of the cost of getting sick and the probability of disease transmission from a sick person to a healthy person. This function satisfies Common Ranking, Supermodularity, and DWTP. We again assume a constant returns to scale matching function where $1/(1 + \gamma)$ is the elasticity of substitution of M.

Positive Sorting There is a symmetric PAM CSE if and only if $\kappa \leq 2(1+\gamma)$ and $\mathbb{I} = [0, \bar{i}]$ for some $\bar{i} \leq 1$. Note that we have already assumed $\kappa < 4$, which ensures $u(i, i) > 0$ for all i. In general, the functional form of the equilibrium allocation is ugly and not very illuminating. Here we highlight the solution when $\kappa \to 4$. In that case, $U(i) = \bar{\lambda}e^{-2i/(1-2i)}(1 - 2i)$ for $i \in [0, \frac{1}{2}]$ and $U(i) = 0$ for $i > \frac{1}{2}$. The equlibrium matching rate for type i agents is $\bar{\lambda}e^{-2i/(1-2i)}/(1 - 2i)$ for $i \in [0, \frac{1}{2}]$ and 0 for $i > \frac{1}{2}$, so lower types are kept out of the market by rationing meetings, while types above $\frac{1}{2}$ give up on meetings altogether. Fees are equal to zero in all active markets.

Figure 4 depicts a symmetric PAM CSE with less extreme parameter values. The top right panel verifies that with private information, market utility is decreasing (Lemma 3). The middle row shows that higher types of agents have a lower matching probability, while platforms matching them have a higher matching probability. With DWTP, this serves to keep lower (sicker) types out of the market, because they value matching more than higher types do. Additionally, the bottom row shows that fees are zero in all active markets.

We highlight an important difference between these results and the symmetric PAM

[9]The assumption that $\kappa < 4$ ensures $u^s(i, j) > 0$ for all i and j; if this assumption fails, there is a market breakdown for all types $i > \bar{i}$, where $\bar{i}$ is the minimum solution to $u^s(\bar{i}, \bar{i}) = 0$.

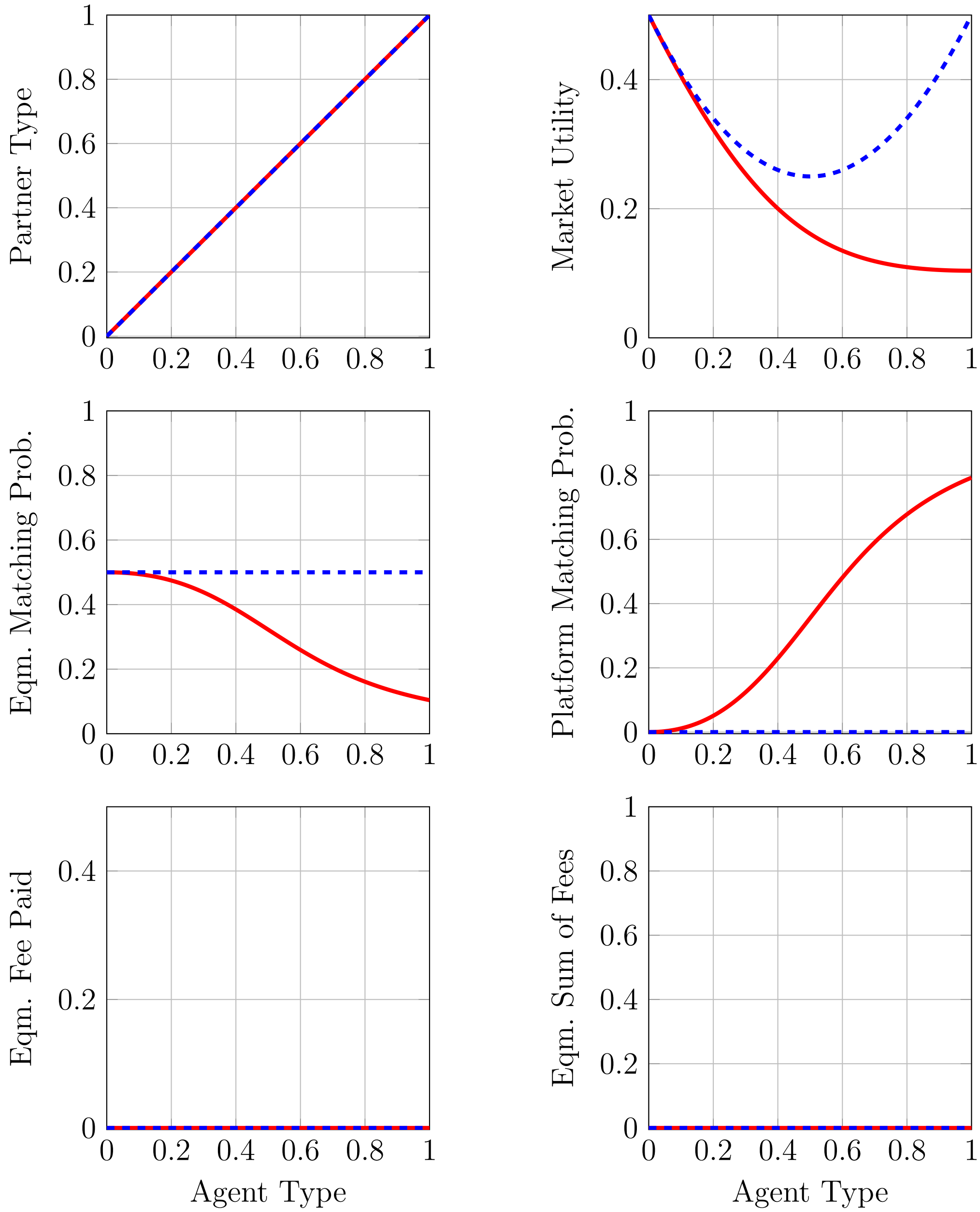

Figure 4: Positively Sorted Equilibrium. Notes: Red solid lines represent private-information equilibrium outcomes, while blue dashed lines show observable-type equilibrium outcomes. Parameters: Parameters: $u(i, j) = 1 - 2i(1 - j)$ and $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$; types are distributed uniformly on $[0, 1]$.

CSE with IWTP. In both cases, platforms earn zero profits. But with IWTP, this happens because they advertised so much that they almost never matched agents. When they did match agents, they earned positive fees. With DWTP, platforms do meet agents with positive probability, but fees are equal to zero.

Cross-Subsidized Matching When we violate the assumptions that $\kappa \leq 2(1+\gamma)$ or that the lowest type is 0, there is no longer a symmetric PAM CSE. In both cases, we run into the same problem: a symmetric PAM CSE pins down market utility. But at this level of market utility, a platform can profitably create a market matching the lowest type to a somewhat higher type. Indeed, this must happen in a CSE.

Figure 5 depicts the resulting cross-subsidized CSE. The basic logic is similar to the IWTP case. In equilibrium, 0.2 matches with 0.27. This boosts their market utility from 0.34 with symmetric information to 0.35. The less healthy type matches with a higher probability, 0.57, while the healthier type matches with complementary probability 0.43, leaving the platform with zero matching probability. Again, these differences in matching probabilities are not a choice, but instead are dictated by the local incentive constraints (Lemma 4): Since u is supermodular, $u_1(0.27, 0.2) < u_1(0.27, 0.27) < 0$. This reduces 0.27's matching probability when matched with the lower type. Compensating for this, the sicker type pays a positive fee, while the healthier type receives the fee as a payment, with the platform collecting no revenue.

This relaxes the incentive constraints for higher types, and hence necessitates rematching them. For example, 0.21 matches with 0.23. This proceeds until 0.22, which matches with its own type. All agents with types above 0.22 also engage in positive assortative matching, so those in the interval $[0.22, 0.27]$ match both with lower types and with agents with the same type.

In summary, with DWTP, we again find that competing platforms effectively cross-subsidize the sickest agents, boosting the market utility of all other types of agents. This cross-subsidization again comes by changing who those agents match with, reinforcing the interplay between private information and sorting.

6.2 Generalization of Results

The examples we just went through suggest two general takeaways: (1) in a CSE, platforms use matching probability to screen agents, while the fees collected sum to zero; and (2) positive sorting requires both sufficient complementarity in the production technology and some special condition to handle the slack information constraint for the lowest type. Here we prove that these results hold more generally. We start with the sum of fees.

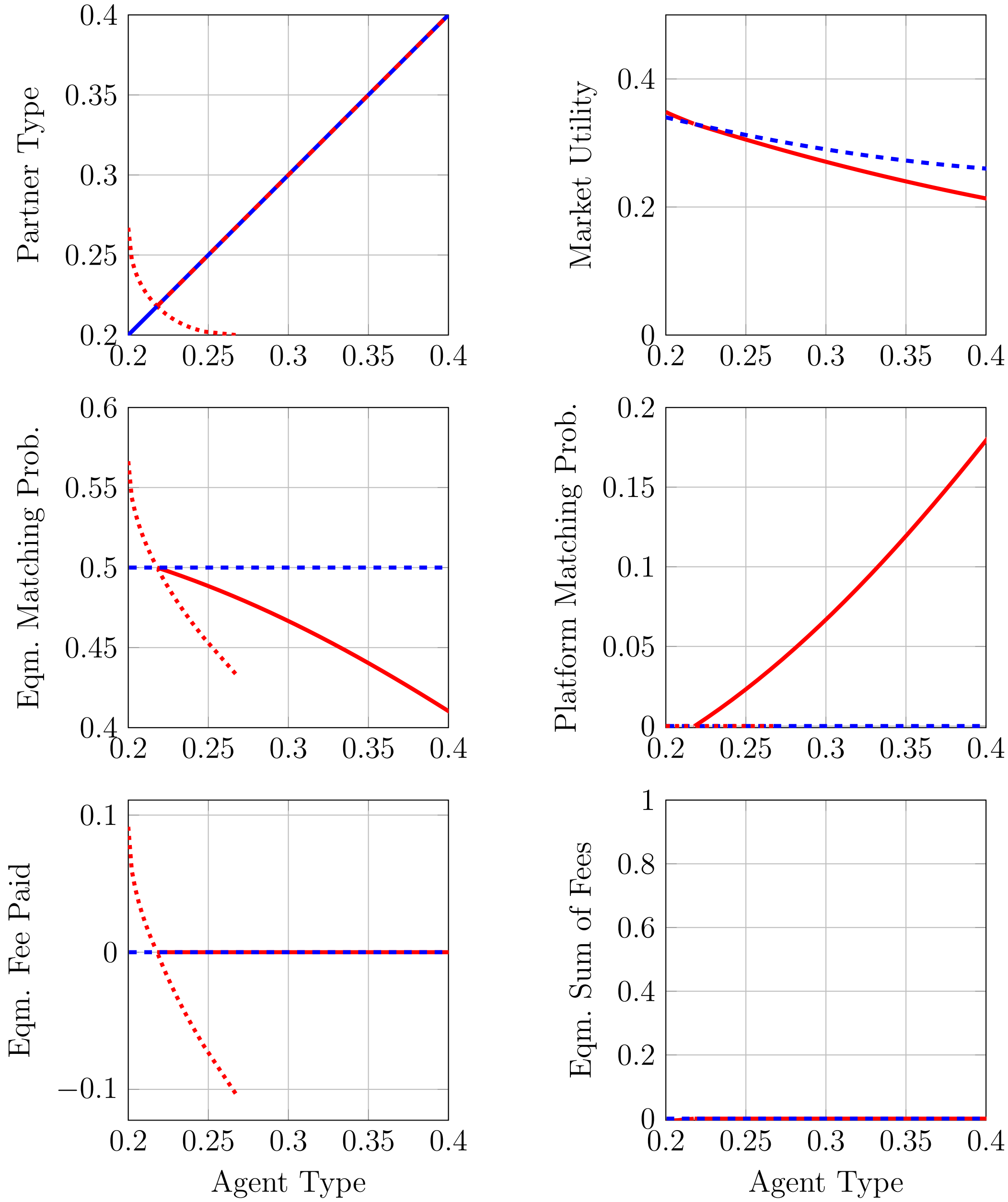

Figure 5: Cross-subsidized Equilibrium. Notes: Red solid and dotted lines represent private-information equilibrium outcomes, while blue dashed lines show observable-type equilibrium outcomes. Parameters: $u(i,j) = 1 - 2i(1-j)$ and $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$; types are distributed uniformly on $[0.2, 0.4]$.

Lemma 6 *Assume Common Ranking, Supermodularity, and DWTP, as well as* $c = 0$. *For any profit-maximizing terms-of-trade* $\tau \in T$ *in a CSE,* $\phi^a + \phi^b = 0$.

The proof is in Appendix C. Intuitively, in a competitive search equilibrium, platforms must break even. If a platform charges a positive sum of fees, that means its matching probability must be zero. So consider a different platform that tries to attract the same types at slightly lower fees. With lower fees, agents are willing to participate in the deviating platform with a lower matching probability, leading to a strictly positive matching probability for the platform. But of course this may not be consistent with equilibrium: the reduction in fees may attract different types. With DWTP, higher types are more willing to accept low fees in return for a low matching probability. This improvement in the types attracted to the terms-of-trade further raises the profitability of the deviating platform by allowing it to charge still higher fees. In this way, the we can find a new terms of trade that has positive fees and a positive matching probability, inconsistent with a CSE.

Once again, the disease example is a special case of a more general class of assumptions:

Assumption 7

(a) $u(i,j) + u(j,i) \leq u(i,i)\frac{u_1(i,j)}{u_1(i,i)} + u(j,j)\frac{u_1(j,i)}{u_1(j,j)}$

(b) For all $i > \underline{i}$, *either* $u(\underline{i},i) + \hat{\phi}(i) \leq 0$ *or* $\left(\bar{\lambda}\frac{u(\underline{i},\underline{i})}{u(\underline{i},i)+\hat{\phi}(i)}, \hat{\ell}(i)\right) \notin \Lambda^o$, *where*

$$\hat{\phi}(i) \equiv u(i,\underline{i}) - \frac{u_1(i,\underline{i})}{u_1(i,i)}u(i,i) \quad \text{and} \quad \hat{\ell}(i) \equiv \bar{\lambda}\frac{u(\underline{i},\underline{i})u_1(i,i)}{u(i,i)u_1(i,\underline{i})}\exp\left(\int_{\underline{i}}^{i} \frac{u_1(k,k)}{u(k,k)}dk\right).$$

Again, Assumption 7(a) ensures that creating a market matching i and j with $i > j > \underline{i}$ is not profitable. In our disease example, this condition is always satisfied. Assumption 7(b) rules out the profitability of a market matching $i > \underline{i}$ with $\underline{i}$.

Proposition 3 *Assume Common Ranking, Supermodularity, DWTP, Symmetric Environment, and Assumption 7. If* $c = 0$, *there is a symmetric PAM CSE. For the terms of trade* $\tau(i)$ *attracting type* i *agents, the matching probability is* $\Lambda(\tau(i)) \equiv \ell(i)$ *solving* $\frac{\ell'(i)}{\ell(i)} = -\frac{u_2(i,i)}{u(i,i)}$, *with* $\ell(0) = \bar{\lambda}$; *the fee is* $\phi^a = \phi^b = 0$; *and market utility solves* $\frac{U'(i)}{U(i)} = \frac{u_1(i,i)}{u(i,i)}$, *with* $U(0) = \bar{\lambda}u(0,0)$.

The proof is in Appendix C and follows the same structure as the proof of Proposition 2. The general characterization is similar to our example: fees are zero, but instead depressed matching probabilities are used to exclude lower types from terms-of-trade intended for higher types.

7 Positive Advertising Costs

We now consider the case where advertising is costly, $c > 0$. For simplicity, we assume a Symmetric Environment and look for a symmetric PAM CSE. We relax the symmetry assumption in Appendix E.1 and consider negative assortative matching in Appendix E.2

We find that both rationing and fees are used in equilibrium, but whether higher types are more or less rationed and pay higher or lower fees depends on the direction of the monotonicity of willingness-to-pay. The intuition for this is straightforward. For a match between two of the lowest type of agents, there is no information problem, since incentive constraints only bind downwards (Corollary 2). Still, there must be some rationing ($\lambda < \bar{\lambda}$) and some fees ($\phi > 0$) so that the platforms can cover their advertising costs. With IWTP, higher types face higher fees for the same reason as in the case without advertising costs: they work to exclude lower types. But if fees are higher, advertising would be more profitable, unless the platform meeting rate falls. And so $m(\lambda, \lambda)$ must be lower for higher types, or equivalently the agents' meeting rate must be higher.

With DWTP, information constraints imply that higher types have a lower matching probability, which means that platforms have a higher matching probability. If fees did not adjust, advertising would be more profitable. Thus competition between platforms must drive down the fees paid by the higher types.

Our formal analysis proceeds constructively. First, we note that any symmetric PAM CSE must solve problem (9), specialized to the case where $k^a = k^b$ and $U^a = U^b$. Moreover, the maximized value of the problem must be c. That is, for all $i \in (\underline{i}, \bar{i}]$, we have

$$\begin{aligned} c = \max_{\lambda \in [0, \bar{\lambda}]} \; & 2m(\lambda, \lambda)\phi \\ \text{s.t.} \;\; & U'(i) = \lambda u_1(i, i), \\ & U(i) = \lambda(u(i, i) - \phi), \\ & \phi \geq 0. \end{aligned}$$

Since $c > 0$, we know that any solution to this problem must have $\phi > 0$. We can thus drop the nonnegativity constraint on ϕ and use the participation constraint to eliminate ϕ from the objective function:

$$\begin{aligned} c = \max_{\lambda \in [0, \bar{\lambda}]} \; & 2m(\lambda, \lambda)\left(u(i, i) - \frac{U(i)}{\lambda}\right) \\ \text{s.t.} \;\; & U'(i) = \lambda u_1(i, i). \end{aligned} \tag{11}$$

If we knew $U(i)$, this problem would uniquely pin down λ. But since we do not know $U(i)$, we instead use this problem to determine it.

First, solve the problem for the lowest type, $i = \underline{i}$, where the local incentive constraint is slack. In this case, we can solve the dual, $U(\underline{i}) = \max_{\lambda \in [0,\bar{\lambda}]} h(\lambda, \underline{i})$, where

$$h(\lambda, i) \equiv \lambda \left(u(i,i) - \frac{c}{2m(\lambda,\lambda)} \right). \tag{12}$$

This pins down $U(\underline{i})$ as well as agents' matching rate in these markets, $\ell(\underline{i}) = \ell^*(\underline{i})$, where in general $\ell^*(i) \equiv \arg\max_{\lambda \in [0,\bar{\lambda}]} h(\lambda, i)$ is the matching rate in an (i,i) market if there were no local incentive constraint. Note that if $c \geq 2m(0,0)u(\underline{i},\underline{i})$, $U(\underline{i}) = 0$. We instead assume in this section that $c < 2m(0,0)u(i,i)$ for all i, so an unconstrained (i,i) market could cover the advertising costs c.

Next, the objective function in problem (11) tells us that if there is a PAM equilibrium, $U(i) = h(\ell(i), i)$, where $\ell(i)$ is the equilibrium value of λ in an (i,i) market. Differentiating this expression with respect to i and using the functional form of h in equation (12) gives us

$$U'(i) = h_1(\ell(i), i)\ell'(i) + \ell(i)\big(u_1(i,i) + u_2(i,i)\big).$$

The local IC constraint states $U'(i) = \ell(i)u_1(i,i)$, so we can simplify this to get

$$h_1(\ell(i), i)\frac{\ell'(i)}{\ell(i)} = -u_2(i,i). \tag{13}$$

This is an ordinary differential equation which we can solve uniquely using the boundary condition $\ell(\underline{i}) = \ell^*(\underline{i})$. This provides a precise set of necessary conditions for any equilibrium with PAM in this environment.

The structure of equation (13) reveals how incentive constraints distort matching probabilities relative to the full information benchmark $\ell^*(i)$. The left-hand side captures the rate of change in matching probabilities, scaled by h_1 which measures the marginal value of increasing the meeting probability. The right-hand side involves $u_2(i,i)$, which captures how an agent's utility changes when their partner's type increases.

The following Proposition combines these insights:

Proposition 4 *Assume Common Ranking, Supermodularity, Monotone WTP, and a symmetric environment. Also assume $2u(i,i)m(0,0) > c > 0$. If there exists a symmetric PAM CSE, then:*

1. *The matching probabilities $\ell(i)$ solve ordinary differential equation (13) with limiting condition $\ell(\underline{i}) = \ell^*(\underline{i})$;*

2. *Market utilities satisfy* $U(i) = h(\ell(i), i)$*;*

3. *Fees satisfy* $\Phi(i) = u(i, i) - U(i)/\ell(i)$*;*

4. *These solutions satisfy all global incentive constraints.*

Moreover, with

- *IWTP,* $U(i)$*,* $\ell(i)$*, and* $\Phi(i)$ *are increasing in* i *and* $\ell(i) \in (\ell^*(i), \bar{\lambda}]$ *for all* $i > \underline{i}$*;*

- *DWTP,* $U(i)$*,* $\ell(i)$*, and* $\Phi(i)$ *are decreasing in* i *and* $\ell(i) \in (0, \ell^*(i))$ *for all* $i > \underline{i}$*.*

The proof is in Appendix D.

The discussion in the text glosses over four technical issues. First, at $i = \underline{i}$, the optimization problem is unconstrained, so the first order condition $h_1(\ell(\underline{i}), \underline{i}) = 0$ holds. This in turn means that we cannot directly apply the ordinary differential equation (13) at $i = \underline{i}$. Instead, we look for a solution to the differential equation with the property that $\lim_{i \to \underline{i}} \ell(i) = \ell^*(\underline{i})$.

Second, Lemma 3 tells us that market value is monotone and so almost everywhere differentiable. This means that the differential equation (13) only holds almost everywhere, leaving open the possibility of discontinuities in ℓ function. The proof of Proposition 4 establishes that ℓ must in fact be continuous.

Third, there are two such solutions, one with ℓ increasing and one with ℓ decreasing. We show in the proof of Proposition 4 that the first solution is valid with IWTP and the second with DWTP.

Lastly, is it possible that the solution to the differential equation (13), combined with the boundary conditions, give us matching probabilities and equilibrium utilities that are non-positive? We show this can never be the case in the proof of Proposition 4. These technical considerations, while subtle, are crucial for a complete understanding of the equilibrium structure. They highlight how the matching technology's properties interact with agents' preferences to determine the appropriate solution concept.

Proposition 4 has several notable implications. First, the differential equation characterization of equilibrium has remarkable simplicity and power. While we only imposed local incentive constraints, the resulting allocation automatically satisfies global incentive constraints. This relies on the way that CSE pins down market utility, which functions as agents' outside option.

Second, the direction of Monotone Willingness-to-Pay completely determines the structure of equilibrium utilities, contact rates, and fees. With IWTP, higher types receive higher utilities and face higher contact rates and fees. With DWTP, the pattern reverses. This stands in contrast to the case with observable types, where the direction of these patterns depends on whether the equilibrium payoff $u(i,i)$ is increasing or decreasing, not on the willingness-to-pay.

Third, private information systematically distorts allocations relative to the observable types benchmark. With IWTP, equilibrium contact rates exceed the full-information optimum $\ell^*(i)$. The free entry condition then implies that fees must also be higher to cover platform costs. With DWTP, these distortions reverse: both contact rates and fees fall below their full-information levels. These distortions reflect platforms' optimal response to incentive constraints: they adjust contact rates and fees to make mimicking less attractive to lower types.

We close this section by noting that, while Proposition 4 provides a complete characterization of symmetric PAM equilibria when they exist, it does not guarantee existence. Once we find candidate equilibrium utilities $U(i)$ by solving the ODE, we know all incentive constraints will be satisfied, both local and global. However, we still must verify that no platform would prefer to create a market matching different types. When $c = 0$, Propositions 2 and 3 provide sufficient conditions for a PAM equilibrium to exist, but we have not derived useful conditions when $c > 0$. Still, since we know market utility in this candidate equilibrium, we just need to check the value of $\hat{V}(k^a, k^b)$ in equation (10), a straightforward numerical computation.

8 Concluding Remarks

How do individuals sort in the presence of private information? We use a competitive search model to answer that question. We prove that under Common Ranking and Supermodularity, any profit maximizing terms of trade only attracts one type of agent to each side of the market. We then characterize PAM under those two assumptions, monotone Willingness-to-Pay, and symmetry. When advertising costs are zero and there is IWTP, only fees differ across active markets, and the matching rate is driven to the highest feasible value. With DWTP, only matching rates differ across markets, and fees are driven to zero. With positive advertising costs, whether we have IWTP or DWTP determines whether fees and matching rates are increasing or decreasing in the agents' type.

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Appendix

A Proofs for Section 4

Proof of Lemma 1. First, we verify that the construction yields a partial equilibrium. We check each condition of Definition 1:

1. (Optimal Search) Take any $\tau \in T^p$. By construction:
   (a) $U^s(i) \geq \bar{U}^s(i, \tau, \Lambda^s(\tau))$ for all $i \in \mathbb{I}^s$, since this is exactly the first constraints of Problem (7);
   (b) $\int_{\mathbb{I}^s} U^s(i)dG^s(i) = \int_{\mathbb{I}^s} \bar{U}^s(i, \tau, \Lambda^s(\tau))dG^s(i)$, from the second constraints.

2. (Impossible Terms-of-Trade) For any $\tau \notin T^p$, by construction there exist no $(\lambda^s)_{s=a,b} \in \mathbb{\Lambda}$ satisfying the constraints of Problem (7). This directly implies there are no $(\lambda^s)_{s=a,b}$ satisfying conditions 2(a) and 2(b) of Definition 1

3. (Profit Maximization) Take any $\tau \in T^p$ with $\tau \notin T$. By construction, $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ does not solve Problem (7). Therefore, there exists some $(\hat{\lambda}^s, \hat{\phi}^s, \hat{G}^s)_{s=a,b}$ satisfying the constraints with strictly higher objective value. Let $\hat{\tau} = (\hat{\phi}^s, \hat{G}^s)_{s=a,b}$, so $\Lambda^s(\hat{\tau}) = \hat{\lambda}^s$. Then $\hat{\tau} \in T^p$ and $V(\hat{\tau}, \Lambda^a(\hat{\tau}), \Lambda^b(\hat{\tau})) > V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$

For the converse, take any partial equilibrium $\{T^p, T, (\Lambda^s, U^s)_{s=a,b}\}$ and any $\tau = (\phi^s, G^s)_{s=a,b} \in T$. We must show that $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ solves Problem (7). From Definition 1 parts 1(a) and 1(b), this tuple satisfies the constraints of Problem (7). Suppose it is not optimal. Then there exists some $(\hat{\lambda}^s, \hat{\phi}^s, \hat{G}^s)_{s=a,b}$ satisfying the constraints with strictly higher objective value. Let $\hat{\tau} = (\hat{\phi}^s, \hat{G}^s)_{s=a,b}$. The constraints of Problem (7) imply $\hat{\tau} \in T^p$ with $\Lambda^s(\hat{\tau}) = \hat{\lambda}^s$. But then $V(\hat{\tau}, \Lambda^a(\hat{\tau}), \Lambda^b(\hat{\tau})) > V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$, so part 3 of Definition 1 implies $\tau \notin T$, a contradiction. ■

Proof of Lemma 2. Suppose not, so U^s is discontinuous at some point $i^* \in [0, 1]$. Then there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exist a point i with $|i - i^*| < \delta$ and $|U^s(i) - U^s(i^*)| > \epsilon$.

First assume it is possible to find such a point with $U^s(i^*) - U^s(i) > \epsilon$. By the third condition in the definition of competitive search equilibrium, type i^* must obtain utility $U^s(i^*) = \Lambda^s(\tau)(\int u^s(i^*, j)dG^{\bar{s}}(j) - \phi^s)$ in some market with terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b} \in T$.

Continuity of u^s in its first argument implies we can choose δ sufficiently small such that $\int u^s(i^*,j)dG^{\bar{s}}(j) - \int u^s(i,j)dG^{\bar{s}}(j) < \epsilon$. Since $0 \leq \Lambda^s(\tau) \leq 1$, this means

$$U^s(i) < U^s(i^*) - \epsilon = \Lambda^s(\tau)\left(\int u^s(i^*,j)dG^{\bar{s}}(j) - \phi^s\right) - \epsilon < \Lambda^s(\tau)\left(\int u^s(i,j)dG^{\bar{s}}(j) - \phi^s\right).$$

The first inequality comes from the assumed discontinuity in U^s. The equation is the indifference condition of i^*. The second inequality uses the continuity of u^s. But this implies that the terms-of-trade τ do not satisfy condition 1(a) in the definition of partial equilibrium for the type i agent, a contradiction.

If instead we have $U^s(i) - U^s(i^*) > \epsilon$, we reverse the role of i and i^* in the proof, but the argument is otherwise unchanged. ■

Proof of Proposition 1.

Necessity. We first prove that any CSE $\left(T^p, T, (\Lambda^s, U^s)_{s=a,b}, \mu\right)$ must satisfy these conditions. Condition 4 is a piece of the definition of CSE. Condition 3 holds in any partial equilibrium, and a CSE is a partial equilibrium. Thus Condition 3 holds as well.

Next, take any $\tau \in T$. Let $\lambda^s = \Lambda^s(\tau)$. Lemma 1 implies that $(\lambda^s, \phi^s, G^s)_{s=a,b}$ must solve problem (7) with maximal value c (from the free entry condition in the definition of CSE). We prove the solution to problem (7) must have a degenerate distribution G^s and moreover must solve problem (8).

Suppose to the contrary that either G^a or G^b is non-degenerate. Let k_1^s be the largest s-side agent with a binding incentive constraint:

$$k_1^s = \max\left\{i \in \mathbb{I}^s | U^s(i) = \lambda^s\left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j)dG^{\bar{s}}(j) - \phi^s\right)\right\} \tag{14}$$

u^s is continuous in its first argument by assumption, and U^s is continuous in a CSE (Lemma 2). Then since $\mathbb{I}^s$ is compact, the type k_1^s exists. Moreover, k_1^s exceeds all points in the support of G^s, strictly so for a positive measure on at least one side given non-degeneracy.

Define new fees $\phi_1^s \equiv u^s(k_1^s, k_1^{\bar{s}}) - U^s(k_1^s)/\lambda^s$. Common ranking implies $\phi_1^s \geq \phi^s$ with strict inequality on at least one side, proving $\phi_1^a + \phi_1^b > \phi^a + \phi^b$.

We next prove that for any $k' < k_1^s$, $U^s(k') \geq \lambda^s(u^s(k', k_1^{\bar{s}}) - \phi^s)$:

$$\begin{aligned}
\lambda^s(u^s(k', k_1^{\bar{s}}) - \phi_1^s) &= \lambda^s \int_{\mathbb{I}^{\bar{s}}} (u^s(k', k_1^{\bar{s}}) - u^s(k', j))dG^{\bar{s}}(j) \\
&\qquad\qquad + \lambda^s \int_{\mathbb{I}^{\bar{s}}} u^s(k', j)dG^{\bar{s}}(j) - \lambda^s \phi_1^s \\
&\leq \lambda^s \int_{\mathbb{I}^{\bar{s}}} (u^s(k_1^s, k_1^{\bar{s}}) - u^s(k_1^s, j))dG^{\bar{s}}(j) \\
&\qquad\qquad + \lambda^s \int_{\mathbb{I}^{\bar{s}}} u^s(k', j)dG^{\bar{s}}(j) - \lambda^s \phi_1^s \\
&= \lambda^s \phi_1^s - \lambda^s \phi^s + \lambda^s \int_{\mathbb{I}^{\bar{s}}} u^s(k', j)dG^{\bar{s}}(j) - \lambda^s \phi_1^s \\
&\leq U^s(k').
\end{aligned}$$

The first equality adds and subtracts $\int_{\mathbb{I}^{\bar{s}}} u^s(k', j)dG^{\bar{s}}(j)$ and reorders terms. The first inequality uses supermodularity, together with $k' < k_1^s$ and $k_1^{\bar{s}} \geq j$ for all j in the support of $G^{\bar{s}}$. The second equality uses the definition of ϕ_1^s as well as the binding constraint in the definition of k_1^s in equation (14). The final inequality collects terms and uses the incentive constraint in the original problem, $U^s(k') \geq \lambda^s\big(\int_{\mathbb{I}^{\bar{s}}} u^s(k', j)dG^{\bar{s}}(j) - \phi^s\big)$.

We do not claim $U^s(k') \geq \lambda^s(u^s(k', k_1^{\bar{s}}) - \phi^s)$ for $k' > k_1^s$. Instead, we construct monotone sequences (k_n^s, ϕ_n^s) by iteratively finding the highest type willing to pay more to match with previous highest types. Suppose for some $n \geq 2$, we have found non-decreasing sequences of types $k_1^s \leq \cdots \leq k_{n-1}^s$ as well as sequences of fees $\phi_1^s, \ldots, \phi_{n-1}^s$ with the following properties:

1. $U^s(k_{n-1}^s) = \lambda^s\big(u^s(k_{n-1}^s, k_{n-1}^{\bar{s}}) - \phi_{n-1}^s\big)$, $s \in \{a, b\}$;
2. $U^s(k') \geq \lambda^s\big(u^s(k', k_{n-1}^{\bar{s}}) - \phi_{n-1}^s\big)$ for all $k' < k_{n-1}^s$, $k' \in \mathbb{I}^s$, $s \in \{a, b\}$;
3. $\phi_{n-1}^a + \phi_{n-1}^b \geq \phi_1^a + \phi_1^b$.

We have done this when $n = 2$ and proceed by induction to extend this to any n.

Let

$$k_n^s \equiv \max\left(\arg\max_{k' \in \mathbb{I}^s}\left(u^s(k', k_{n-1}^s) - \frac{U^s(k')}{\lambda^s}\right)\right). \tag{15}$$

That is, k_n^s is the largest element of the set of maximizers of $u^s(k', k_{n-1}^s) - U^s(k')/\lambda^s$. The first two properties above imply that for all $k' < k_{n-1}^s$,

$$u^s(k', k_{n-1}^{\bar{s}}) - \frac{U^s(k')}{\lambda^s} \leq \phi_{n-1}^s = u^s(k_{n-1}^s, k_{n-1}^{\bar{s}}) - \frac{U^s(k_{n-1}^s)}{\lambda^s}.$$

Thus $k_n^s \geq k_{n-1}^s$. Also define

$$\phi_n^s \equiv u^s(k_n^s, k_n^{\bar{s}}) - \frac{U^s(k_n^s)}{\lambda^s}$$

The same arguments as for $n = 1$ imply $\phi_n^s \geq \phi_{n-1}^s$ and $U^s(k') \geq \lambda^s(u^s(k', k_n^{\bar{s}}) - \phi_n^s)$ for all $k' < k_n^s$.

The sequences (k_n^a, k_n^b) are nondecreasing on the compact set $\mathbb{I}^a \times \mathbb{I}^b$ and so converge to $(k^{a*}, k^{b*}) \in \mathbb{I}^a \times \mathbb{I}^b$. This means the ϕ^s converges as well, to (ϕ^{a*}, ϕ^{b*}), with $\phi^{a*} + \phi^{b*} \geq \phi_1^a + \phi_1^b > \phi^a + \phi^b$. Continuity of U^s and u^s implies

1. $U^s(k^{s*}) = \lambda^s\big(u^s(k^{s*}, k^{\bar{s}*}) - \phi^{s*}\big)$, $s \in \{a, b\}$;

2. $U^s(k') \geq \lambda^s\big(u^s(k', k^{\bar{s}*}) - \phi^{s*}\big)$ for all $k' < k^{s*}$, $k' \in \mathbb{I}^s$, $s \in \{a, b\}$;

3. $\phi^{a*} + \phi^{b*} \geq \phi_1^a + \phi_1^b > \phi^a + \phi^b$.

Finally, we claim the second point, the incentive constraint, extends to all $k' \in \mathbb{I}^s$, including $k' > k^{s*}$. To prove this, suppose to the contrary that there is a $k' > k^{s*}$ with $U^s(k') < \lambda^s\big(u^s(k', k^{\bar{s}*}) - \phi^{s*}\big)$. Eliminate ϕ^{s*} using the first bullet point:

$$U^s(k') < \lambda^s\big(u^s(k', k^{\bar{s}*}) - u^s(k^{s*}, k^{\bar{s}*})\big) + U^s(k^{s*}).$$

But continuity of u^s and convergence of $k_n^{\bar{s}}$ to $k^{\bar{s}*}$ implies that there exists an n such that

$$U^s(k') < \lambda^s\big(u^s(k', k_n^{\bar{s}}) - u^s(k^{s*}, k_n^{\bar{s}})\big) + U^s(k^{s*}),$$

or equivalently

$$u^s(k', k_n^{\bar{s}}) - \frac{U^s(k')}{\lambda^s} > u^s(k^{s*}, k_n^{\bar{s}}) - \frac{U^s(k^{s*})}{\lambda^s},$$

From the definition of k_{n+1}^s (equation (15)), this implies $k' > k^{s*}$, which in turn contradicts monotonicity of the sequence k_n^s with convergence to k^{s*}. So we conclude that $U^s(k') \geq \lambda^s\big(u^s(k', k^{\bar{s}*}) - \phi^{s*}\big)$ for all $k' \in \mathbb{I}^s$, $s \in \{a, b\}$.

Now let G^{s*} be the cumulative distribution function that is degenerate at k^{s*} and consider the alternative term-of-trade $(\lambda^s, \phi^{s*}, G^{s*})_{s=a,b}$. By construction this satisfies the constraints in problem (7) and it attains a higher value than the original term-of-trade because $\phi^{a*} + \phi^{b*} > \phi^a + \phi^b$ and (λ^a, λ^b) is unchanged. This proves that the original term-of-trade does not solve problem (7). That is, any solution to the problem (7) must have a degenerate type distribution on both sides of the market.

This proves the first two conditions in the statement of the Lemma, since any solution to problem (7) must have a degenerate type distribution and slack upward constraints, and thus also solve problem (8). Following the same steps, we also find that any solution to problem (8) also solves problem (7).

Sufficiency. For sufficiency, we need to prove that a tuple satisfying the four conditions constitutes both a partial equilibrium and a CSE. Let's verify each requirement:

First, for the partial equilibrium conditions from Definition 1:

1. Optimal Search and Impossible Terms-of-Trade follow directly from condition 3;

2. For Profit Maximization, we proceed in steps. Condition 1 ensures each $\tau \in T$ is separating, and condition 2 ensures it solves problem (8). Given the equivalence between problems (7) and (8) when condition 1 is satisfied, Lemma 1 allows us to map from the solution to problem (8) back to a partial equilibrium. Thus condition 2 ensures that each $\tau \in T$ satisfies the final partial equilibrium requirement, profit maximization.

Next, for the additional CSE requirements from Definition 2:

1. Free Entry is satisfied because condition 2 ensures each $\tau \in T$ achieves value exactly equal to c;

2. Market Clearing follows directly from condition 4.

Thus, the tuple constitutes a CSE. ∎

Proof of Lemma 3. This result follows directly from the incentive constraints in problem (7).

Choose any s-side type-i. If $U^s(i) > 0$, the market for these agents clears in a CSE. That is, there is an equilibrium terms-of-trade that attracts them, with matching probability λ^s, fee ϕ^s, and partner distribution $G^{\bar{s}}(j)$. From problem (7):

$$U^s(i) = \lambda^s \left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j) dG^{\bar{s}}(j) - \phi^s \right)$$

Conversely, all other types i' weakly prefer not to come to this market

$$U^s(i') \geq \lambda^s \left(\int_{\mathbb{I}^{\bar{s}}} u^s(i',j) dG^{\bar{s}}(j) - \phi^s \right).$$

Subtracting inequalities gives us

$$U^s(i') - U^s(i) \geq \lambda^s \int_{\mathbb{I}^{\bar{s}}} \big(u^s(i',j) - u^s(i,j) \big) dG^{\bar{s}}(j).$$

Now we use the monotone WTP assumptions. With IWTP, the right hand side of this last inequality is nonnegative when $i' > i$, so U^s is nondecreasing. With DWTP, the right hand side is nonnegative when $i' < i$, so U^s is nonincreasing.

Finally, if $U^s(i) = 0$, nonnegativity of U^s implies $U^s(i') \geq U^s(i)$ for all i', ensuring weak monotonicity in this case as well. ■

Proof of Lemma 4. From Lemma 3, Monotone Willingness-to-Pay implies that market utilities $U^s(i)$ are monotone. By Lebesgue's differentiation theorem, any monotone function is differentiable almost everywhere.

Now fix a $\tau \in T$, with $\lambda^s = \Lambda^s(\tau)$ and G^s degenerate at k^s for $s \in \{a, b\}$. Also assume $k^a \in (0, 1)$, so that for all small positive ϵ, $0 \leq k^a - \epsilon < k^a + \epsilon \leq 1$. The constraints from problem (7) require

$$
\begin{aligned}
U^a(k^a + \epsilon) - U^a(k^a) &\geq \lambda^a(u^a(k^a + \epsilon, k^b) - u^a(k^a, k^b)), \\
U^a(k^a - \epsilon) - U^a(k^a) &\geq \lambda^a(u^a(k^a - \epsilon, k^b) - u^a(k^a, k^b)),
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{U^a(k^a + \epsilon) - U^a(k^a)}{\epsilon} &\geq \lambda^a \frac{(u^a(k^a + \epsilon, k^b) - u^a(k^a, k^b))}{\epsilon}, \\
\frac{U^a(k^a) - U^a(k^a - \epsilon)}{\epsilon} &\leq \lambda^a \frac{(u^a(k^a, k^b) - u^a(k^a - \epsilon, k^b))}{\epsilon}.
\end{aligned}
$$

Since U^a is differentiable almost everywhere, it must be $U^{a\prime}(k^a) = \lambda^a u_1^a(k^a, k^b)$ at such points.

A symmetric argument establishes the result for side b. ■

B Proofs for Section 5

Proof of Lemma 5. We prove this lemma by contradiction. Suppose, to the contrary, $m(\Lambda^a(\tau), \Lambda^b(\tau)) > 0$ for a terms-of-trade $\tau \in T$. Suppose τ attracts type i on the side a and j on the side b using fees (ϕ^a, ϕ^b). Since $c = 0$ and $m(\Lambda^a(\tau), \Lambda^b(\tau)) > 0$, free entry requires $\phi^a + \phi^b = 0$. WLOG, assume $\phi^a \leq 0$.

From Part 1 of the definition of a partial equilibrium (optimal search), it must be the case that

$$
\Lambda^a(\tau) = \frac{\bar{U}(i)}{u^a(i, j) - \phi^a} \leq \frac{\bar{U}(i')}{u^a(i', j) - \phi^a}
$$

for all i'. Now consider a small increase in ϕ^a to $\phi' < u^a(i, j)$. Holding fixed the b side of the market, this higher fee will attract a new type i' at a new contact rate λ' satisfying

$$
\lambda' = \frac{\bar{U}(i')}{u^a(i', j) - \phi'} \leq \frac{\bar{U}(i)}{u^a(i, j) - \phi'}.
$$

If $\phi' - \phi^a$ is sufficiently small, we still have $m(\lambda', \Lambda^b(\tau)) > 0$. Multiplying inequalities and canceling the $\bar{U}$ terms gives us

$$(u^a(i,j) - \phi^a)(u^a(i',j) - \phi') \geq (u^a(i',j) - \phi^a)(u^a(i,j) - \phi')$$
$$\Rightarrow (\phi' - \phi^a)(u^a(i',j) - u^a(i,j)) \geq 0.$$

Since $\phi' > \phi^a$ and u^a is increasing in its first argument, this implies $i' \geq i$. If $i' = i$, we are done: the terms of trade $\tau' = (\phi', \phi^b, i, j)$ attains contact rates $\lambda' < \Lambda^a(\tau)$ and $\Lambda^b(\tau)$ yielding positive profits, a contradiction.

If $i' > i$, the terms of trade τ' is impossible. Instead, we try to create a new market matching i' to j. Since $i' > i$, we can raise the fee on the b side of the market while holding the matching rate λ^b fixed. As in the proof of Proposition 1, this will inductively lead to a new terms-of-trade that is possible, has positive fees, and has a platform matching rate $m(\lambda', \Lambda^b(\tau)) > 0$, a contradiction. ■

Proof of Proposition 2.

Characterizing a PAM CSE. To start, we show that under the proposed equilibrium utilities, $U(0) = \bar{\lambda} u(0,0)$ and $U'(i) = \bar{\lambda} u_1(i,i)$, all i solve a symmetric version of the optimization problem (9):

$$\max_{\lambda \in [0,\bar{\lambda}]} 2m(\lambda,\lambda)\left(u(i,i) - \frac{U(i)}{\lambda}\right) \tag{16}$$
$$\text{s.t. } U(j) - U(i) \geq \lambda(u(j,i) - u(i,i)) \; \forall j < i, j \in \mathbb{I},$$
$$u(i,i) - \frac{U(i)}{\lambda} \geq 0.$$

We also prove the maximized value is zero. Recall that $\bar{\lambda}$ is the highest matching probability that can be obtained by both sides of the market under symmetry, so $m(\bar{\lambda}, \bar{\lambda}) = 0$.

The first constraint in problem (16) evaluated at j slightly smaller than i implies $\lambda \geq \bar{\lambda}$ when evaluated at the market utility:

$$U'(i) = \lim_{j \to i} \frac{U(j) - U(i)}{j - i} \leq \lambda \lim_{j \to i} \frac{u(j,i) - u(i,i)}{j - i} = \lambda u_1(i,i),$$

where we reverse the inequality because we have divided both sides by $j - i < 0$. Since $U'(i) = \bar{\lambda} u_1(i,i)$ and $u_1(i,i) > 0$, this implies $\lambda \geq \bar{\lambda}$. The requirement that $m(\lambda,\lambda) \geq 0$ implies $\lambda \leq \bar{\lambda}$. Thus given the level of market utility $U(i)$, the only feasible matching rate is $\bar{\lambda}$ if there is to be an active (i,i) market.

Next we prove that the second constraint problem (16) is satisfied. Again use $U(0) = \bar{\lambda}u(0,0)$ and $U'(i) = \bar{\lambda}u_1(i,i)$. Since $u_2(i,i) > 0$, this implies $U'(i) < \bar{\lambda}(u_1(i,i) + u_2(i,i))$. Hence for $i > 0$, $\bar{\lambda}u(i,i) > U(i)$, so the second constraint is satisfied. This implies that the fee charged in this market is positive, $\Phi(i) = u(i,i) - \frac{U(i)}{\bar{\lambda}} > 0$; equivalently, $\Phi(0) = 0$ and $\Phi'(i) = u_2(i,i) > 0$.

Verifying Downward ICs. So far in characterizing the equilibrium outcome, we have only incorporated the local incentive constraints. This step verifies that the equilibrium outcomes indeed satisfy all the global downward incentive constraints. More precisely:

$$U(j) - U(i) \geq \ell(i)(u(j,i) - u(i,i)), \quad \forall j \leq i.$$

To do so, we use the characterization of the market utility to write

$$U(j) - U(i) = -\bar{\lambda}\int_j^i u_1(x,x)dx \geq -\bar{\lambda}\int_j^i u_1(x,i)dx = \bar{\lambda}(u(j,i) - u(i,i)),$$

where in the first equality we use $U'(i) = \bar{\lambda}u_1(i,i)$, in the inequality we use supermodularity of $u(i,j)$, and in the last equality we solve out the integral. This proves the global ICs also hold on the PAM CSE.

Existence of a PAM CSE. We still need to rule out the possibility that an (i,j) market is more profitable for $i \neq j$. First, assume that i and j are both positive, so in a proposed (i,j) market, we have a binding local incentive constraint on both sides:

$$U'(i) = \lambda^a u_1(i,j), \; U'(j) = \lambda^b u_1(j,i).$$

Using $U'(i) = \bar{\lambda}u_1(i,i)$ and $U'(j) = \bar{\lambda}u_1(j,j)$, we can write this as

$$\lambda^a = \bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \;\; \lambda^b = \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}.$$

By assumption, however, $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}\right) \notin \mathbb{A}^o$, and any combination of such matching probabilities is infeasible.

Alternatively, if $j = 0 < i$, then there is only a binding incentive constraint on one side of the market: $U'(i) = \lambda^a u_1(i,0)$. Assumption 6 (b) rules the profitable deviations to $(i,0)$

markets in steps. First, if $u_1(k,0)=0$ for all k, then for some $k<i$

$$U(k)-U(i)=-\bar{\lambda}\int_k^i u_1(x,x)dx < -\bar{\lambda}\int_k^i u_1(x,0)dx = 0,$$

In the first equality, we use the characterization of the PAM CSE; in the first inequality, we use the supermodularity; and in the last equality, we evaluate the integral. This means for any λ^a, $U(k)-U(i)<0<\lambda^b(u(k,0)-u(i,0))$ violating the incentive constraint for type k in the $(i,0)$ market. Thus the $(i,0)$ market is infeasible.

Otherwise, we can find λ^a that satisfies the downward ICs. Inverting the participation constraints, we have the sum of fees on either side of the market must be

$$\phi^a+\phi^b = u(0,i)+u(i,0)-\frac{\bar{\lambda}u(0,0)}{\lambda^b}-\frac{u_1(i,0)}{u_1(i,i)}\left(u(0,0)+\int_0^i u_1(k,k)dk\right)\lambda^a \le u(0,i)+\tilde{\phi}(i)$$

where in the inequality we use the definition of $\tilde{\phi}(i)$ and the fact $\frac{\bar{\lambda}u(0,0)}{\lambda^b}\ge 0$. Thus, if $u(0,i)+\tilde{\phi}(i)\le 0$, there are no terms-of-trade that deliver a positive sum of fees to the platform, ruling out the profitable deviations to $(i,0)$ markets.

Otherwise, there is (λ^a,λ^b) that satisfies the incentive constraints and delivers a positive sum of fees. When $\lambda^b=\bar{\lambda}\frac{u(0,0)}{u(0,i)+\phi}$, the sum of fees is zero, and the sum of fees is strictly positive (negative) if $\lambda^b>(<)\bar{\lambda}\frac{u(0,0)}{u(0,i)+\phi}$. If $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,0)},\bar{\lambda}\frac{u(0,0)}{u(0,i)+\tilde{\phi}(i)}\right)\notin \mathbb{A}^o$, then any pair of matching probability that delivers positive sum of fees is infeasible from the matching function, ruling out the profitable devaition to $(i,0)$ market.

CES Case. Assume a CES payoff function:

$$u(i,j)=\left(\frac{1}{2}i^{\frac{\theta-1}{\theta}}+\frac{1}{2}j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}.$$

Also assume a CES matching function

$$m(\lambda^a,\lambda^b)=\left(1-(\lambda^a)^\gamma-(\lambda^b)^\gamma\right)^{\frac{1}{\gamma}}.$$

This implies $\mathbb{A}=\{(\lambda^a,\lambda^b)|(\lambda^a)^\gamma+(\lambda^b)^\gamma\le 1\}$, and $\bar{\lambda}=2^{-1/\gamma}$.

Note that in this case, $u(i,i)=i$ and $u_1(i,i)=u_2(i,i)=\frac{1}{2}$ for all i. This implies $U(i)=\bar{\lambda}i/2$ and $\Phi(i)=i/2$ for all i. That is, if a PAM equilibrium exists, half the gains from trade are lost because of the private information problem.

For the CES case, the condition $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}\right) \notin \mathbb{A}$ can be expressed as

$$\frac{1}{2}i^{\frac{\gamma}{\theta}} + \frac{1}{2}j^{\frac{\gamma}{\theta}} > \left(\frac{1}{2}i^{\frac{\theta-1}{\theta}} + \frac{1}{2}j^{\frac{\theta-1}{\theta}}\right)^{\frac{\gamma}{\theta-1}}$$

Alternatively, let $x = i^{\frac{\theta-1}{\theta}}$ and $y = j^{\frac{\theta-1}{\theta}}$. Then we require

$$\frac{1}{2}x^{\frac{\gamma}{\theta-1}} + \frac{1}{2}y^{\frac{\gamma}{\theta-1}} > \left(\frac{1}{2}x + \frac{1}{2}y\right)^{\frac{\gamma}{\theta-1}}$$

Using Jensen's inequality, this holds for all $x \neq y$ if and only if $\gamma > \theta - 1$. Conversely, if $\theta > 1 + \gamma$, there is no PAM equilibrium because creating any other (i, j) market would be profitable.

Now consider a $(0, j)$ market. If $\theta \leq 1$, $u_1(j, 0) = 0$ for all j and so incentive constraints require $\lambda^b = \lim_{i\to 0} \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)} = +\infty$. This means it is never feasible to create a $(0, j)$ market. For $\theta > 1$,

$$\lambda^b = \bar{\lambda}\frac{u_1(j,j)}{u_1(j,0)} = 2^{\frac{1+\gamma-\theta}{\gamma(\theta-1)}}.$$

This is larger than 1, meaning it is impossible to create such a market, whenever $\theta < 1 + \gamma$. This proves that for $\theta < 1 + \gamma$, there is a PAM CSE, while for larger values of θ such an equilibrium cannot exist. ■

C Proofs for Section 6

Proof of Lemma 6. We prove this lemma by contradiction. Suppose, to the contrary, $\phi^a + \phi^b > 0$ for a terms-of-trade $\tau \in T$. Suppose τ attracts type i on the side a and j on the side b and has matching probabilities $(\Lambda^a(\tau), \Lambda^b(\tau))$. Since $c = 0$ and $\phi^a + \phi^b > 0$, free entry requires $m(\Lambda^a(\tau), \Lambda^b(\tau)) = 0$. WLOG, assume $\Lambda^a(\tau) > 0$.

From Part 1 of the definition of a partial equilibrium (optimal search), it must be the case that

$$\Lambda^a(\tau) = \frac{\bar{U}(i)}{u^a(i,j) - \phi^a} \leq \frac{\bar{U}(i')}{u^a(i',j) - \phi^a}$$

for all i'. Now consider a small decrease in ϕ^a to ϕ'. Holding fixed the b side of the market, this higher fee will attract a new type i' at a new contact rate $\lambda' < \Lambda^a(\tau)$ satisfying

$$\lambda' = \frac{\bar{U}(i')}{u^a(i',j) - \phi'} \leq \frac{\bar{U}(i)}{u^a(i,j) - \phi'}.$$

If $\phi^a - \phi'$ is sufficiently small, we still have $\lambda' > 0$. Multiplying inequalities and canceling the $\bar{U}$ terms gives us

$$(u^a(i,j) - \phi^a)(u^a(i',j) - \phi') \geq (u^a(i',j) - \phi^a)(u^a(i,j) - \phi')$$
$$\Rightarrow (\phi' - \phi^a)(u^a(i',j) - u^a(i,j)) \geq 0.$$

Since $\phi' < \phi^a$ and u^a is decreasing in its first argument, this implies $i' \geq i$.

The remainder of the proof is identical to the proof of Lemma 5. If $i' = i$, we are done: the terms of trade $\tau' = (\phi', \phi^b, i, j)$ attains contact rates $\lambda' < \Lambda^a(\tau)$ and $\Lambda^b(\tau)$ yielding positive profits, a contradiction. If $i' > i$, the terms of trade τ' is impossible. Instead, we try to create a new market matching i' to j. Since $i' > i$, we can raise the fee on the b side of the market while holding the matching rate λ^b fixed. As in the proof of Proposition 1, this will inductively lead to a new terms-of-trade that is possible, has positive fees, and has a platform matching rate $m(\lambda', \Lambda^b(\tau)) > 0$, a contradiction. ■

Proof of Proposition 3. We follow the structure of the proof of Proposition 2.

Characterization. To start, we show that under the proposed equilibrium utilities, the equilibrium matching probabilities, fees, and utilities satisfy the proposed formulas. To start, we return to the optimization problem given equilibrium utilities:

$$\max_{\lambda \in [0,\bar{\lambda}]} 2m(\lambda,\lambda)\left(u(i,i) - \frac{U(i)}{\lambda}\right) \tag{17}$$
$$\text{s.t. } U(j) - U(i) \geq \lambda(u(j,i) - u(i,i)) \; \forall j < i, j \in \mathbb{I},$$
$$u(i,i) - \frac{U(i)}{\lambda} \geq 0.$$

With DWTP, we can divide through the first set of constraints to get $\lambda \leq \frac{U(j)-U(i)}{u(j,i)-u(i,i)}$ for $j < i$. We want to show that $\Phi(i) = 0$ for any i. Suppose, to the contrary, the sum of fees is strictly positive. Then the platform can deviate to another terms-of-trade, charging the same set of fees while offering a lower matching probability on side a. Since the incentive constraint puts an upper bound on matching probabilities, this deviation is feasible. Because the matching probability of the platform decreases with λ^a, the proposed deviation delivers positive fees with a positive matching probability, which contradicts the optimization of the platforms. We then rule out the possibilities of $\Phi(i) < 0$. For the fees For $\Phi(i) < 0$ in the equilibrium, it must $\ell(i) = \bar{\lambda}$. Suppose this is indeed the case; type 0 would be strictly better off deviating to such terms-of-trade. More precisely, by participating to such a (i,i) market, the type 0 receives a payoff of:

$$\bar{\lambda}(u(0,i) - \Phi(i)) > \bar{\lambda}u(0,i) > \bar{\lambda}u(0,0) = U(0),$$

where the first inequality uses the assumption $\Phi(i) < 0$, and the second inequality uses common ranking. This violates the IC. Thus, for $i > 0$, $\Phi(i) = 0$.

Next, zero fees imply that $U(i) = \ell(i)u(i,i)$. Totally differentiating this condition, we have $U'(i) = \ell'(i)u(i,i) + \ell(i)(\partial_1 u(i,i) + \partial_2 u(i,i))$. Imposing the IC $U'(i) = \ell(i)u_1(i,i)$, we have $\ell'(i)u(i,i) = -\ell(i)u_2(i,i)$. This differential equation, with the initial condition $\ell(0) = \bar{\lambda}$, has a solution:

$$\ell(i) = \bar{\lambda}\exp\left(-\int_0^i \frac{u_2(\iota,\iota)}{u(\iota,\iota)}d\iota\right) > 0.$$

Thus we can divide the condition $U'(i) = \bar{\lambda}u_1(i,i)$ by $U(i) = \bar{\lambda}u(i,i)$ to get $\frac{U'(i)}{U(i)} = \frac{u_1(i,i)}{u(i,i)}$.

Verifying Downward ICs. We verify that

$$U(j) - U(i) \geq \ell(i)(u(j,i) - u(i,i)), \ \forall j \leq i$$

From the characterization of the equilibrium

$$\begin{aligned} & U(j) - U(i) \\ =& -\int_j^i \ell(x)u_1(x,x)dx \geq -\int_j^i \ell(i)u_1(x,x)dx \geq -\ell(i)\int_j^i u_1(x,i)dx \\ =& \ell(i)(u(j,i) - u(i,i)), \end{aligned}$$

where in the first equality we use the local ICs on the PAM CSE, in the first inequality we used $u_1(x,x) \leq 0$ and $\ell(i)$ is monotonically decreasing in i, in the second inequality we used $-\ell(i) < 0$ and the supermodularity of $u(i,j)$, and in the last equality we evaluate the integral. This shows the global downward ICs hold.

Sufficient Conditions We look for sufficient conditions for a PAM CSE. We conjecture that there is a PAM CSE and look at a market which attracts type i agents on side a and type j agents on side b. If all such markets are unprofitable or infeasible, then there is a PAM CSE. To start, we assume that both $i > 0$ and $j > 0$.

In the conjectured market, the local incentive constraints require

$$\lambda^a = \frac{U'(i)}{u_1(i,j)},\ \lambda^b = \frac{U'(j)}{u_1(j,i)}.$$

Inverting the participation for type i and type j, we get:

$$\begin{aligned}\phi^a + \phi^b &= u(i,j) + u(j,i) - \frac{U(i)}{U'(i)} u_1(i,j) - \frac{U(j)}{U'(j)} u_1(j,i) \\ &= u(i,j) + u(j,i) - u(i,i)\frac{u_1(i,j)}{u_1(i,i)} - u(j,j)\frac{u_1(j,i)}{u_1(j,j)},\end{aligned}$$

where in the second equality, we used the solution from PAM CSE. Under assumption 7(a), $\phi^a + \phi^b < 0$. Thus, there is no terms-of-trade attracting (i,j) and delivering non-negative payoffs to platforms.

Now consider a terms-of-trade attracting $(0,i)$, where $i > 0$. With supermodularity and decreasing WTP, it must be $u_1(i,0) < 0$ for any $i > 0$, otherwise $u_1(i,j) > 0$ for $j > i$.

We start with the case $0 < i < 1$. It is thus always possible to invert the local incentive constraint to get $\lambda^b = \frac{U'(i)}{u_1(i,0)}$. From the participation constraint of type i in this proposed $(0,i)$ market, the fee charged is

$$\hat{\phi}(i) = u(i,0) - \frac{U(i)}{\lambda^b} = u(i,0) - \frac{u_1(i,0)}{u_1(i,i)} u(i,i)$$

If $\hat{\phi}(i) + u(0,i) \leq 0$, the maximal sum of fees that can be charged by the platform is non-positive, meaning there is no profitable deviation to the $(0,i)$ market.

Now we consider the case $i = 1$. This case is special because the incentive constraint only puts an upper bound on the matching probability

$$\lambda^b \leq \frac{U'(i)}{u_1(i,0)}$$

For such terms-of-trade to be incentive compatible, the matching probability on the a side has to satisfy

$$U(k) - U(i) \geq \lambda^a (u(k,0) - u(i,0)),\ \forall k < j,$$

Suppose $u_1(i,0) = 0$ for any i:

$$U(k) - U(i) = -\int_k^i \ell(x) u_1(x,x) dx \geq -\ell(0) \int_k^i u_1(x,x) dx > -\ell(0) \int_k^i u_1(x,0) dx = 0,$$

where in the first inequality we use $\ell(i)$ is decreasing in i, in the second inequality we use supermodularity, and in the last equality we use $u_1(i,0) = 0$.

Suppose otherwise, we can then use the local incentive constraint to find $\lambda^a = \frac{U'(i)}{u_1(i,0)}$.

The participation constraints imply that for such terms-of-trade, the sum of fees must be

$$\phi^a + \phi^b = u(0,i) + u(i,0) - u(0,0)\frac{\bar{\lambda}}{\lambda^b} - u(i,i)\frac{u_1(i,0)}{u_1(i,i)} \leq u(0,i) + u(i,0) - u(i,i)\frac{u_1(i,0)}{u_1(i,i)} = \hat{\phi}(i)$$

because $u(0,0)\frac{\bar{\lambda}}{\lambda^b} \geq 0$. If $\hat{\phi}(i) \leq 0$, there is no such terms-of-trade that delivers a positive sum of fees to the platforms. Otherwise, the fees are zero when $\lambda^b = \frac{\bar{\lambda}u(0,0)}{\hat{\phi}(i)}$, strictly positive (negative) when $\lambda^b > (<)\frac{\bar{\lambda}u(0,0)}{\hat{\phi}(i)}$. When $\left(\hat{\lambda}(i), \bar{\lambda}\frac{u(0,0)}{\hat{\phi}(i)}\right) \notin \mathbb{\Lambda}^o$, any matching probabilities that deliver a positive sum of fees is infeasbile under the matching function.

Disease Model Solving the differential equations characterizing the PAM equilibrium outcomes, we obtain the market utility

$$U(i) = 2^{-\frac{1}{\gamma}} e^{-\zeta(i)} \sqrt{1 - \kappa i(1-i)},$$

where

$$\zeta(i) \equiv \sqrt{\frac{\kappa}{4-\kappa}} \left(\arctan\left(\sqrt{\frac{\kappa}{4-\kappa}} \right) - \arctan\left(\sqrt{\frac{\kappa}{4-\kappa}}(1-2i) \right) \right).$$

We now verify the sufficient conditions for the PAM equilibrium to exist. The sufficient condition that rules out (i,j) markets, where $i \neq j$ and $i, j > 0$, requires

$$u(i,j) + u(j,i) - u(i,i)\frac{u_1(i,j)}{u_1(i,i)} - u(j,j)\frac{u_1(j,i)}{u_1(j,j)} = -\frac{(i-j)^2}{(1-i)(1-j)} < 0.$$

In the first equality, we imposed the solution from the equilibrium characterization, and the inequality always holds since $i, j \in [0,1]$. Thus, condition (a) of assumption 7 does not impose any additional restrictions on utility.

Now we move to ruling out $(0,i)$ markets with $i > 0$. Under a PAM equilibrium, the local IC on side b of this market requires that

$$\ell^b(i) = \frac{U'(i)}{u_1(j,0)} = 2^{-\frac{1}{\gamma}} \frac{1-i}{\sqrt{1-\kappa i(1-i)}} e^{-\zeta(i)}.$$

Using this, we can recover the fees on side b. When the sum of fees is zero, we then get the fees on side a. Finally, the participation constraint on side a give us $\ell^a(i)$:

$$\ell^a(i) = 2^{-\frac{1}{\gamma}} \frac{1-i}{1-2i}.$$

For λ^a to be a positive number, it has to be $i \in \left[0, \frac{1}{2}\right)$. We want to find conditions such that

$T(i) \equiv 1 - (\ell^a(i))^\gamma - (\ell^b(i))^\gamma \le 0$, for $i \in \left[0, \frac{1}{2}\right)$. From the definition, $T(0) = 0$. We want to show that $T'(i) \le 0$ for $i \in \left[0, \frac{1}{2}\right)$. Taking this derivative, we can show

$$sign\left(T'(i)\right) = sign\left((1 - \kappa i\,(1-i))^{-\frac{\gamma}{2}-1} \exp\left(-\gamma\zeta\,(i)\right) - (1-2i)^{-\gamma-1}\right)$$

Thus $T'(i) \le 0$ is equivalent to

$$H(i) \equiv -\gamma\zeta(i) + (1+\gamma)\log(1-2i) + \left(1 + \frac{\gamma}{2}\right)\log(1 - \kappa i(1-i)) \le 0$$

We can also show that $H(0) = 0$. Taking derivative

$$H'(i) = -\gamma\zeta'(i) - (1+\gamma)\frac{2}{1-2i} + \left(1+\frac{\gamma}{2}\right)\frac{\kappa(1-2i)}{1-\kappa i(1-i)}$$

Using the definition of $\zeta(i)$ we have $\zeta'(i) = -\frac{\kappa}{2(1-\kappa i(1-i))}$. Thus:

$$H'(i) = \frac{2\kappa i^2 - \kappa(2-\gamma)i - 2(1+\gamma) + \kappa}{(1-2i)(1-\kappa i(1-i))}$$

To determine the sign of $H'(i)$, we focus on the sign of the numerator, a quadratic function. Because $2\kappa > 0$, the maximum of the numerator is reached either at $i = 0$ or $i = \frac{1}{2}$. At $i = 0$, $H'(0) = \kappa - 2(1+\gamma)$; At $i = \frac{1}{2}$, $H'\left(\frac{1}{2}\right) = \frac{1}{2}(1+\gamma)(\kappa - 4)$. When $\kappa \le \min\{2(1+\gamma), 4\}$, $H'(i) \le 0$ for any $i \in \left[0, \frac{1}{2}\right)$. Since $H(0) = 0$, this shows that $H(i) \le 0$ (and $T'(i) \le 0$) for all $i \in \left[0, \frac{1}{2}\right)$. Since $T(0) = 0$, this also proves the condition b in assumption (7) is met when $\kappa \le \min\{2(1+\gamma), 4\}$.

■

D Proof for Section 7

Proof of Proposition 4. We seek to characterize an (i,i) market in a competitive search equilibrium in the symmetric case. If such an equilibrium exists, then for all $i = k^a = k^b$, the maximized value of Problem (9) must be c:

$$\begin{aligned} c = \max_{\lambda \in [0,\bar{\lambda}]} & \; 2m(\lambda,\lambda)\left(u(i,i) - \frac{U(i)}{\lambda}\right) \\ \text{s.t. } & U(j) - U(i) \ge \lambda(u(j,i) - u(i,i)) \text{ for all } j < i \\ & U(i) \le \lambda u(i,i) \end{aligned}$$

Moreover, since $c > 0$, any solution must satisfy the second constraint, which we thus drop:

$$c = \max_{\lambda \in [0,\bar{\lambda}]} 2m(\lambda,\lambda)\left(u(i,i) - \frac{U(i)}{\lambda}\right) \tag{18}$$
$$\text{s.t. } U(j) - U(i) \geq \lambda(u(j,i) - u(i,i)) \text{ for all } j < i.$$

Unconstrained Markets. First suppose the incentive constraints are all slack for some i. Using equation (4), we can rewrite the unconstrained problem in terms of the number of agents per posting, $n = N(\lambda,\lambda)$:

$$c = \max_{n \geq 0} 2\big(M^*(n)u(i,i) - nU(i)\big) \tag{19}$$

where $M^*(n) \equiv M(n,n)$. Then since M is strictly concave, the objective has a unique maximum satisfying $M^{*\prime}(n)u(i,i) = U(i)$. Plugging this into the fact that the objective function must equal c, $c = 2\big(M^*(n)u(i,i) - nU(i)\big)$, we get

$$\frac{c}{2u(i,i)} = M^*(n) - nM^{*\prime}(n)$$

Again using concavity of the matching function M, the right hand side is increasing. It is equal to zero at $n = 0$ and, since $2u(i,i)m(0,0) > c$, it exceeds $c/(2u(i,i))$ as n grows without bound.[10] Thus n is uniquely defined if an unconstrained (i,i) market exists in equilibrium; call this value $n^*(i)$.

Once we know $n = n^*(i)$, we can recover $U(i) = U^*(i)$ from the free entry condition: $c = 2\big(M^*(n^*(i))u(i,i) - n^*(i)U^*(i)\big)$. We can also recover the matching probability $\ell^*(i) = L(n^*(i), n^*(i))$ from equation (3). Lastly, we can recover $\Phi^*(i)$ from $U^*(i) = \ell^*(i)(u(i,i) - \Phi^*(i))$. Because $c > 0$, we must have positive matching probabilities and positive fees, $m(\ell(i),\ell(i)) > 0$ and $\Phi^*(i) > 0$.

If there is a competitive search equilibrium with PAM, this determines the matching probability and agent value when $i = 0$, since there are no incentive constraints in the $(0,0)$ market. For other markets, the constraints may bind.

Constrained Markets. Now suppose at least one of the incentive constraints in problem (18) is binding, i.e. the unconstrained solution violates an incentive constraint. Since the objective function is decreasing in $U(i)$, if $U(i) > U^*(i)$ the objective evaluates to less than c for all values of λ. This is inconsistent with an active (i,i) market in a compet-

[10] It is straightforward to prove from the definitions of M^*, m, n, and λ that $\lim_{n\to\infty} M^*(n) - nM^{*\prime}(n) = m(0,0)$.

itive search equilibrium. We thus conclude that $U(i) \leq U^*(i)$ in any competitive search equilibrium.

If $U(i) = U^*(i)$, the free entry condition $c = 2\big(M^*(n)u(i,i) - nU(i)\big)$ pins down $n = n^*(i)$, which by assumption violates an incentive constraint. So we must have $U(i) < U^*(i)$. In this case, there are two values of n that satisfy the free entry condition

$$c = 2\big(M^*(n)u(i,i) - nU(i)\big).$$

This is because (i) the right hand side is zero when $n = 0$; (ii) the right hand side is negative for a sufficiently large value of n since the slope of $M^*(n)$ is less than 1 and decreasing for all positive n; (iii) the right hand side is strictly increasing and strictly larger than c when $n = n^*(i)$ since $U(i) < U^*(i)$; and (iv) the right hand side is a strictly concave function of n. This implies that there are two solutions to this equation, one in the interval $[0, n^*(i))$ and the other in $(n^*(i), \infty)$.

Now since $M^*(n)$ is concave with $M^*(0) = 0$, agents' matching rate $M^*(n)/n$ is a decreasing function of n. Thus we can equivalently state that for $U(i) < U^*(i)$, there are two solutions $\lambda \in [0, \bar{\lambda}]$ to the equation

$$c = 2m(\lambda, \lambda)\left(u(i,i) - \frac{U(i)}{\lambda}\right).$$

One of the solutions is in the interval $(\ell^*(i), \bar{\lambda}]$, while the other lies in the interval $(0, \ell^*(i))$. Which solution is relevant depends on whether we have IWTP or DWTP, and so we treat those cases separately.

IWTP. First, suppose there is IWTP. The incentive constraint in Problem (18), which only holds for $j < i$, implies a lower bound on λ:

$$\lambda \geq \sup_{j<i} \frac{U(i) - U(j)}{u(i,i) - u(j,i)}.$$

Now if the solution to problem (18) has $\ell(i)$ equal to the smaller solution, i.e. $\ell(i) \in (0, \ell^*(i))$, then $\ell^*(i)$ is feasible and, since $U(i) < U^*(i)$, attains a value in problem (18) that exceeds c, a contradiction. So it must be that the solution to problem (18) has $\ell(i)$ equal to the larger solution, $\ell(i) \in (\ell^*(i), \bar{\lambda}]$, whenever $U(i) < U^*(i)$ in a competitive search equilibrium in the IWTP case.

Next, we replace the global incentive constraints in Problem (18) with local ones:

$$c = \max_{\lambda \in (\ell^*(i), \bar{\lambda}]} 2m(\lambda, \lambda)\left(u(i,i) - \frac{U(i)}{\lambda}\right)$$
$$\text{s.t. } U'(i) = \lambda u_1(i,i).$$

There is nothing to optimize here. Instead, these equations define the path of $U(i)$ and $\ell(i)$:

$$c = 2m(\ell(i), \ell(i))\left(u(i,i) - \frac{U(i)}{\ell(i)}\right), \quad U'(i) = \ell(i)u_1(i,i), \quad \text{and } \ell(i) \in (\ell^*(i), \bar{\lambda}]. \tag{20}$$

Since $U(i)$ is continuous (Lemma 2), the first equation and the restriction that $\ell(i) \in (\ell^*(i), \bar{\lambda}]$ ensures $\ell(i)$ is continuous. Therefore from the second equation $U'(i)$ is continuous.

Now solve the first equation in (20) for $U(i)$, differentiate with respect to i, and eliminate $U'(i)$ using the second equation. This gives us equation (13) in the text and establishes that it must hold at all i, since $\ell(i)$ is continuous and $U(i)$ is continuously differentiable.

We next prove monotonicity. We know $U(i)$ is increasing from the local incentive constraint $U'(i) = \ell(i)u_1(i,i)$. Because $2u(0,0)m(0,0) > c$, $U(0) > 0$, and monotonicity of $U(i)$ implies that $U(i) > 0$ for $i > 0$. To prove $\ell(i)$ is increasing, we first establish that for fixed i, h in equation (12) is single-peaked in $\lambda \in [0, \bar{\lambda}]$, achieving maximum value $U^*(i)$ at $\ell(i) = \ell^*(i)$, the unconstrained case. To prove this, note that we have already established that the objective function in problem (18) is single-peaked: for all $U(i) < U^*(i)$, there exists thresholds λ_1 and λ_2 with $\lambda_1 < \ell^*(i) \leq \lambda_2$ such that

$$c < 2m(\lambda, \lambda)\left(u(i,i) - \frac{U(i)}{\lambda}\right) \Leftrightarrow \lambda \in (\lambda_1, \lambda_2).$$

Flipping this inequality around, we get that

$$U(i) < \lambda\left(u(i,i) - \frac{c}{2m(\lambda, \lambda)}\right) = h(\lambda, i) \Leftrightarrow \lambda \in (\lambda_1, \lambda_2).$$

This proves that $h(\lambda, i)$ is single-peaked. Moreover, setting $U(i) = U^*(i)$ and using a similar logic implies that $\max_{\lambda \in [0, \bar{\lambda}]} h(\lambda, i) = h(\ell^*(i), i) = U^*(i)$.

Since $\ell(i) \geq \ell^*(i)$ in the IWTP case, $h_1(\ell(i), i) < 0$. Then the differential equation (13) implies $\ell'(i) > 0$, proving ℓ is increasing. Finally, to show that $\Phi(i)$ is increasing, we totally differentiate the constraint $U(i) = \ell(i)(u(i,i) - \Phi(i))$:

$$U'(i) = \ell(i)(u_1(i,i) + u_2(i,i) - \Phi'(i)) + \ell'(i)(u(i,i) - \Phi(i)).$$

The local incentive constraint requires that $U'(i) = \ell(i)u_1(i,i)$, so we can write:

$$\Phi'(i) = u_2(i,i) + \frac{\ell'(i)}{\ell(i)^2}U(i)$$

The right-hand side is positive because of common ranking, $\ell'(i) \geq 0$, and $U(i) > 0$. Thus $\Phi(i)$ must be increasing as well.

DWTP. With DWTP, the logic is reversed. The constraints in Problem (18) imply an upper bound the meeting rate:

$$\lambda \leq \sup_{j<i} \frac{U(j) - U(i)}{u(j,i) - u(i,i)}.$$

We can use that to rule out the upper solution, which is dominated by $\ell^*(i)$. We thus conclude that the solution to problem (18) has $\ell(i)$ equal to the smaller solution, $\ell(i) \in [0, \ell^*(i))$, whenever $U(i) < U^*(i)$ in a competitive search equilibrium in the DWTP case.

We again use the local incentive constraints to get a version of equation (20):

$$c = 2m(\ell(i),\ell(i))\left(u(i,i) - \frac{U(i)}{\ell(i)}\right), \quad U'(i) = \ell(i)u_1(i,i), \quad \text{and } \ell(i) \in [0, \ell^*(i)). \tag{21}$$

Again, we use continuity of $U(i)$ to prove continuity of $\ell(i)$ and then use that to prove continuity of $U'(i)$. Finally, the same algebraic manipulations give us equation (13) in the text and establishes that it must hold at all i.

The monotonicity proof is also reversed. We know $U(i)$ is decreasing from the local incentive constraint $U'(i) = \ell(i)u_1(i,i)$. To prove $\ell(i)$ is decreasing, we use the same property of h in equation (12). Now private information leads to $\ell(i) < \ell(i^*)$ in the DWTP case, implying $h_1(\ell(i), i) < 0$. Then the differential equation (13) implies $\ell'(i) < 0$, proving ℓ is decreasing.

Finally, since $\ell(i)$ is decreasing, $m(\ell(i),\ell(i))$ is increasing and so the free-entry condition $c = 2m(\ell(i),\ell(i))\Phi(i)$ implies $\Phi(i)$ is decreasing. ∎

E Extensions to Baseline Model

This appendix first provides a system of ODEs to describe a PAM CSE without symmetry. We then provide a similar analysis for a NAM CSE. In both cases, our characterization is based on solving an ordinary differential equation system to find the proposed equilibrium, and then checking whether market utility is consistent with the proposed equilibrium. For notational simplicity, we assume throughout that the type distribution has support $\mathbb{I}^s = [0,1]$.

E.1 Positive Assortative Matching without Symmetry

In this section, we discuss an algorithm to compute a PAM CSE without imposing that the environment is symmetric. For exposition simplicity, we focus on cases where all types participate: the condition (2) in the definition of a competitive search equilibrium holds with equality for types. We discuss the other cases briefly at the end of this section.

Our characterization follows two steps. First, in a PAM equilibrium, for the type 0's market to clear, $\sigma(0) = 0$. That is, some platforms post terms-of-trade that attracts the lowest types on both sides. From Proposition 1, this terms-of-trade is undistorted. Lemma 7 summarizes the details of this terms-of-trade. Secondly, we solve other equilibrium terms-of-trade as solution of a system of odes, detailed in Proposition 5.

Lemma 7 *Assume Common Ranking and Supermodularity. In any PAM CSE, $\sigma(0) = 0$ and $(\lambda^s(0))_{s=a,b}, U^b(0)$ solve the following unconstrained problem, for a fixed $U^a(0) = \upsilon$:*

$$0 = \max_{\lambda^a,\lambda^b} m(\lambda^a,\lambda^b)\left(u^a(0,0) + u^b(0,0) - \frac{\upsilon}{\lambda^a} - \frac{U^b(0)}{\lambda^b}\right), \tag{22}$$

with the solution denoted as $(\lambda_0^{s*}(\upsilon))_{s=a,b}$ and $U_0^{b*}(\upsilon)$.

Lemma 8 *Assume common ranking, supermodularity, and monotone willingness-to-pay. In any PAM CSE, $\lambda^s(i)$ is continuous.*

The proof is in Appendix F.

We characterize the PAM equilibrium using a system of odes in (λ^a, λ^b), summarized in Proposition 5.

Proposition 5 *Assume common ranking, supermodularity and continuous type distribution. A PAM CSE is characterized by $(\lambda^s, U^s)_{s=a,b}$, σ that solve*

$$\nu^a(i)\frac{\ell^{a\prime}(i)}{\ell^a(i)} = -\iota(i) - u_2^b(\sigma(i), i), \qquad \nu^b(i)\frac{\ell^{b\prime}(i)}{\ell^b(i)} = \iota(i) - \sigma'(i)u_2^a(i,\sigma(i)), \tag{23}$$

$$U^{a\prime}(i) = \ell^a(i)u_1^a(i,\sigma(i)), \qquad U^{b\prime}(\sigma(i)) = \ell^b(i)u_1^b(\sigma(i), i), \tag{24}$$

$$\sigma'(i) = \frac{\ell^a(i)}{\ell^b(i)}\frac{I^a\bar{g}^a(i)}{I^b\bar{g}^b(\sigma(i))}, \tag{25}$$

with the boundary conditions

$$\sigma(0)=0,\ \sigma(1)=1,\ \lambda^s(0)=\lambda_0^{s*}(U^a(0)),U^b(0)=U_0^{b*}(U^a(0)),$$

where

$$\nu^a(i)=\left(\frac{\epsilon_1(\ell^a(i),\ell^b(i))}{m(\ell^a(i),\ell^b(i))}c+\frac{U^a(i)}{\ell^a(i)}\right),\ \nu^b(i)=\left(\frac{\epsilon_2(\ell^a(i),\ell^b(i))}{m(\ell^a(i),\ell^b(i))}c+\frac{U^b(\sigma(i))}{\ell^b(i)}\right), \tag{26}$$

$$\iota(i)=\sigma'(i)\nu^a(i)\frac{u^a_{1,2}(i,\sigma(i))}{u^a_1(i,\sigma(i))}-\nu^b(i)\frac{u^b_{1,2}(\sigma(i),i)}{u^b_1(\sigma(i),i)},$$

$$\epsilon_1(\lambda^a,\lambda^b)=\frac{\lambda^a m_1(\lambda^a,\lambda^b)}{m(\lambda^a,\lambda^b)},\ \ \epsilon_2(\lambda^a,\lambda^b)=\frac{\lambda^b m_2(\lambda^a,\lambda^b)}{m(\lambda^a,\lambda^b)}.$$

The proof is in Appendix F.

Discussion. It is possible that there is no solution to the differential equation system from Proposition 5 where $U^s(i)\geq 0$ for all $i\in\mathbb{I}^s$. In this case, the PAM equilibrium involves some types not participating. Under increasing willingness-to-pay, there are three possible non-participating scenarios. First, it could be that the Side-a low types are not participating: there is a threshold type $\underline{i}^a$ such that $U^a(i)=0$ for all $i\leq\underline{i}^a$. In this case, the assignment function starts from $\underline{i}^a$. The second case involves the same pattern, on side b. Lastly, it is possible that the lowest types on both sides do not participate. This happens when none of the terms-of-trade that attract side-s type-0 covers the advertising costs.

Finally, the differential equations in Proposition 5 only characterize a PAM CSE if one exists. One still has to compute the value of creating an (i,j) market, as in equation (10), and verify that this does not exceed the advertising cost for any pair (i,j).

E.2 Negative Assortative Matching

Next, we discuss an algorithm to compute a CSE with negative assortative matching (NAM), where the higher types match with lower types. First, we focus on characterization of the $(0,1)$ market. In this market, the incentive constraint on side-a is irrelevant. Given the market utility $U^b(1)$ and λ^b, the matching proability for side-a type-0 and her market utility are solutions to:

$$c=\max_{\lambda^a} m\left(\lambda^a,\lambda^b\right)\left(u^a(0,1)+u^b(1,0)-\frac{U^a(0)}{\lambda^a}-\frac{U^b(1)}{\lambda^b}\right). \tag{27}$$

For other types, we can solve an ordinary differential equation system:

Proposition 6 *Assume common ranking, supermodularity and continuous type distribution. A NAM CSE is characterized by* $(\lambda^s, U^s)_{s=a,b}$*,* σ *that solve*

$$\nu^a(i)\frac{\ell^{a\prime}(i)}{\ell^a(i)} = -\iota(i) - u_2^b(\sigma(i), i), \qquad \nu^b(i)\frac{\ell^{b\prime}(i)}{\ell^b(i)} = \iota(i) - \sigma'(i)u_2^a(i, \sigma(i)), \tag{28}$$

$$U^{a\prime}(i) = \ell^a(i)u_1^a(i, \sigma(i)), \qquad U^{b\prime}(\sigma(i)) = \ell^b(i)u_1^b(\sigma(i), i), \tag{29}$$

$$\sigma'(i) = -\frac{\ell^a(i)}{\ell^b(i)}\frac{I^a\bar{g}^a(i)}{I^b\bar{g}^b(\sigma(i))}, \tag{30}$$

with the boundary conditions

$$\sigma(0) = 1,\ \sigma(1) = 0,\ (\lambda^s(0), U^s(0)) \text{ solve problem (27) given } (\lambda^{\bar{s}}(1), U^{\bar{s}}(1)).$$

where

$$\nu^a(i) = \left(\frac{\epsilon_1(\ell^a(i), \ell^b(i))}{m(\ell^a(i), \ell^b(i))}c + \frac{U^a(i)}{\ell^a(i)}\right),\ \nu^b(i) = \left(\frac{\epsilon_2(\ell^a(i), \ell^b(i))}{m(\ell^a(i), \ell^b(i))}c + \frac{U^b(\sigma(i))}{\ell^b(i)}\right), \tag{31}$$

$$\iota(i) = \sigma'(i)\nu^a(i)\frac{u_{1,2}^a(i, \sigma(i))}{u_1^a(i, \sigma(i))} - \nu^b(i)\frac{u_{1,2}^b(\sigma(i), i)}{u_1^b(\sigma(i), i)},$$

$$\epsilon_1(\lambda^a, \lambda^b) = \frac{\lambda^a m_1(\lambda^a, \lambda^b)}{m(\lambda^a, \lambda^b)},\ \ \epsilon_2(\lambda^a, \lambda^b) = \frac{\lambda^b m_2(\lambda^a, \lambda^b)}{m(\lambda^a, \lambda^b)}.$$

F Proofs for Section E

Proof of Lemma 8. We start by defining the objective function of the platforms in terms of (n^a, n^b): $W(n^a, n^b) \equiv M(n^a, n^b)\left(u^a(0,0) + u^b(0,0)\right) - U^a(0)n^a - U^b(0)n^b$. As M is strictly concave, there is a unique maximizer to $W(n^a, n^b)$ given any $U^a(0)$ and $U^b(0)$. Correspondingly, there is a unique combination $(\lambda^a(0), \lambda^b(0))$. In a PAM CSE, the market clearing condition requires that $\lim_{i\to 0^+}\sigma(i) = 0$. Otherwise, there is $j \in \mathbb{I}^b$ whose market does not clear.

Now we prove the continuity of $\lambda^s(i)$ by contradition. Suppose, to the contrary of the lemma, $\lambda^s(i)$ is not continuous at 0. Let $l^s = \lim_{i\to 0^+}\lambda^s(i)$. For $i > 0$,

$$c = \hat{V}(i, \sigma(i)) = m(\ell^a(i), \ell^b(i))\left(u(i, \sigma(i)) + u^b(\sigma(i), i) - \frac{U^a(i)}{\ell^a(i)} - \frac{U^b(\sigma(i))}{\ell^b(i)}\right)$$

Taking $i \to_+ 0$:

$$c = \lim_{i \to_+ 0} \hat{V}(i, \sigma(i)) = m(l_0^a, l_0^b) \left(u^a(0,0) + u^b(0,0) - \frac{U^a(0)}{l_0^a} - \frac{U^b(0)}{l_0^b} \right).$$

If $(l^a, l^b) \neq (\lambda^a(0), \lambda^b(0))$, we have found another maximiser for the platform payoff at $(0,0)$. A contradiction. ■

Proof of Proposition 5. It is convenient to use the following notations, where we write the payoff of platforms as a function of $(\lambda^s, k^s)_{s=a,b}$

$$\tilde{V}(\lambda^a, \lambda^b, k^a, k^b) \equiv m(\lambda^a, \lambda^b) \sum_{s=a,b} \left(u^s(k^s, k^{\bar{s}}) - \frac{U^s(k^s)}{\lambda^s} \right),$$

and

$$\hat{\lambda}^s(k^s, k^{\bar{s}}) = \frac{U^{s\prime}(k^s)}{u_1^s(k^s, k^{\bar{s}})}, \text{ for } s = a, b. \tag{32}$$

With this notation:

$$\hat{V}(i,j) = \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j).$$

Among the odes in the lemma, equation (24) comes directly from the local incentive constraints in lemma 4. We start by deriving equation (23).

For a fixed $i \in \mathbb{I}^a$, $\sigma(i)$ has to be the local maximum of $\hat{V}(i,j)$. Otherwise, the platforms can increase their payoffs by choosing $j' \neq \sigma(i)$, violating the optimality condition. Thus, the first-order condition holds $\hat{V}_2(i, \sigma(i)) = 0$. The same argument implies $\hat{V}_1(i, \sigma(i)) = 0$.

Given j, the impact of changing i on $\hat{V}(i,j)$ can be written in terms of three effects: the change in matching probabilities on two sides and the change in the utility from matches:

$$\begin{aligned}
&\frac{\partial \log \hat{V}(i,j)}{\partial i} \\
=&\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial \hat{\lambda}^a(i,j)} \hat{\lambda}_1^a(i,j) + \frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial \hat{\lambda}^b(i,j)} \hat{\lambda}_2^b(j,i) \qquad (33)\\
+&\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial i}. \qquad (34)
\end{aligned}$$

We now unpack the three effects in turns. Starting from the effect through matching probability λ^a. From the definition of $\tilde{V}$:

$$\frac{\partial \log \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}{\partial \lambda^a} = \left(\frac{\epsilon_1(\lambda^a, \lambda^b)}{m(\lambda^a, \lambda^b)} \tilde{V}(\lambda^a, \lambda^b, k^a, k^b) + \frac{U^a(k^a)}{\lambda^a} \right) \frac{m(\lambda^a, \lambda^b)}{\lambda^a \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}.$$

In the PAM CSE, $\tilde{V}(\ell^a(i), \ell^b(i), i, \sigma(i)) = c$, and so we can write

$$\frac{\partial \log \tilde{V}(\ell^a(i), \ell^b(i), i, \sigma(i))}{\partial \ell^a(i)} = \nu^a(i) \frac{m(\ell^a(i), \ell^b(i))}{c\ell^a(i)},$$

where we used the definition of $\nu^a(i)$ as in the lemma. The same logic establishes that $\frac{\partial \log \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}{\partial \lambda^b} = \nu^b(i) \frac{m(\ell^a(i), \ell^b(i))}{c\ell^b(i)}$. To derive $\hat{\lambda}^s_2(i,j)$, we totally differentiate the identity $\ell^a(i) = \hat{\lambda}^a(i, \sigma(i))$ and $\ell^b(i) = \hat{\lambda}^b(\sigma(i), i)$ to get:

$$\ell^{a\prime}(i) = \hat{\lambda}^a_1(i, \sigma(i)) + \hat{\lambda}^a_2(i, \sigma(i))\sigma'(i)$$

$$\ell^{b\prime}(i) = \hat{\lambda}^b_1(\sigma(i), i)\sigma'(i) + \hat{\lambda}^b_2(\sigma(i), i).$$

Differentiating equation (32), we have:

$$\hat{\lambda}^a_2(i, \sigma(i)) = -\ell^a(i) \frac{u^a_{1,2}(i, \sigma(i))}{u^a_1(i, \sigma(i))}, \; \hat{\lambda}^b_2(\sigma(i), i) = -\ell^b(i) \frac{u^b_{1,2}(\sigma(i), i)}{u^b_1(\sigma(i), i)}.$$

So

$$\hat{\lambda}^a_1(i, \sigma(i)) = \ell^{a\prime}(i) + \sigma'(i)\ell^a(i) \frac{u^a_{1,2}(i, \sigma(i))}{u^a_1(i, \sigma(i))},$$

$$\hat{\lambda}^b_1(\sigma(i), i) = \frac{1}{\sigma'(i)} \left(\ell^{b\prime}(i) + \ell^b(i) \frac{u^b_{1,2}(\sigma(i), i)}{u^b_1(\sigma(i), i)} \right).$$

Putting this together, we have

$$\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial \hat{\lambda}^a(i,j)} \hat{\lambda}^a_1(i,j) = \nu^a(i) \frac{m(\ell^a(i), \ell^b(i))}{c} \left(\frac{\ell^{a\prime}(i)}{\ell^a(i)} + \sigma'(i) \frac{u^a_{1,2}(i, \sigma(i))}{u^a_1(i, \sigma(i))} \right),$$

$$\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial \hat{\lambda}^b(i,j)} \hat{\lambda}^b_2(j,i) = \nu^b(i) \frac{m(\ell^a(i), \ell^b(i))}{c} \left(- \frac{u^b_{1,2}(\sigma(i), i)}{u^b_1(\sigma(i), i)} \right).$$

Finally, we directly compute the effect of changing matching partners on utility:

$$\frac{\partial \log \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}{\partial i} = \frac{m(\lambda^a, \lambda^b)}{\tilde{V}(\lambda^a, \lambda^b, k^a, k^b)} \left(u^a_1(k^a, k^b) + u^b_2(k^b, k^a) - \frac{U^{a\prime}(k^a)}{\lambda^a} \right).$$

The incentive constraint (32) and the fact that in the PAM CSE, $\tilde{V}(\ell^a(i), \ell^b(i), i, \sigma(i)) = c$ means that we can write this as

$$\frac{\partial \log \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}{\partial i} = \frac{m(\lambda^a, \lambda^b)}{c} u^b_2(k^b, k^a).$$

Now we have all terms needed to calculate $\frac{\partial \hat{V}(i,j)}{\partial i}$. Setting $\frac{\partial \hat{V}(i,j)}{\partial i} = 0$ and multiplying both sides by $\frac{c}{m(\ell^a(i),\ell^b(i))}$, we get the following equation:

$$0 = \nu^a(i)\left(\frac{\ell^{a\prime}(i)}{\ell^a(i)} + \sigma'(i)\frac{u^a_{1,2}(i,\sigma(i))}{u^a_1(i,\sigma(i))}\right) - \nu^b(i)\frac{u^b_{1,2}(\sigma(i),i)}{u^b_1(\sigma(i),i)} + u^b_2(\sigma(i),i).$$

Similarly, for the b side:

$$0 = -\nu^a(i)\frac{u^a_{1,2}(i,\sigma(i))}{u^a_1(i,\sigma(i))} + \nu^b(i)\frac{1}{\sigma'(i)}\left(\frac{\ell^{b\prime}(i)}{\ell^b(i)} + \frac{u^b_{1,2}(\sigma(i),i)}{u^b_1(\sigma(i),i)}\right) + u^a_2(i,\sigma(i)).$$

Using the definition of $\iota(i)$, we reach the formula as in the lemma.

Now we move on to equation (25). In our model, matches are one-to-one. Thus, $\forall i \in \mathbb{I}^a$, the accumulated numbers of matches on the a side must equal the accumulated numbers of matches on the b side:

$$\int_0^i \lambda^a(i')\bar{g}^a(i')di' = \int_0^{\sigma(i)} \lambda^b(\sigma^{-1}(j'))\bar{g}^b(j')dj'.$$

The equation above has to hold for any i, we differentiate both sides with respect to i:

$$\ell^a(i)\bar{g}^a(i) = \ell^b(i)\bar{g}^b(\sigma(i))\sigma'(i).$$

With the assumption that $\bar{g}^b(j) > 0$ for all $j \in \mathbb{I}^b$, we derive the ode for $\sigma(i)$:

$$\sigma'(i) = \frac{\ell^a(i)}{\ell^b(i)}\frac{I^a\bar{g}^a(i)}{I^b\bar{g}^b(\sigma(i))}.$$

For the market clearing condition to hold for all types, we further require that $\sigma(0) = 0$ and $\sigma(1) = 1$. This conclude the proof. ■

G Model with Observable Types

In this section, we formally define the equilibrium when types are observable and characterize the equilibrium. The core modification we made to the baseline model is a re-definition of terms-of-trade. With observable types, a terms-of-trade is $\tau = (\phi^s, G^s)_{s=a,b}$, which specifies four objects: the type-contingent fees ϕ^a and ϕ^b charged to matched agents on each side, and the distributions G^a and G^b of agent types participating. With observable types, the fees can depend on the types of participating agents, and thus ϕ^s maps the support of type distribution $\mathbb{I}^s$ to the set of real numbers. Let $\mathbb{T}^O$ denote the set of feasible terms-of-trade when types are observable, i.e., the set of such vectors collecting two fee functions and two type distributions. With this definition of terms-of-trade, the value of platforms and agents is accordingly

$$\tilde{V}(\tau, \lambda^a, \lambda^b) \equiv m(\lambda^a, \lambda^b) \left(\int_{\mathbb{I}^a} \phi^a(i) dG^a(i) + \int_{\mathbb{I}^b} \phi^b(i) dG^b(j) \right) \tag{35}$$

and

$$\tilde{U}^s(i, \tau, \lambda^s) \equiv \lambda^s \left(\int_{\mathbb{I}^{\bar{s}}} u^s(i, j) dG^{\bar{s}}(j) - \phi^s(i) \right). \tag{36}$$

The logic closely follows equations (1)–(2), extended to allow for type-specific fees. The definitions of a partial equilibrium and a competitive search equilibrium resemble the ones with private types, even though the fees are type-contingent and the terms-of-trade belong to the set $\mathbb{T}^O$.

Definition 6 *A* ***partial equilibrium with observable types*** $\{T^p, T, (\Lambda^s, U^s)_{s=a,b}\}$ *is two sets* $T \subseteq T^p \subseteq \mathbb{T}^O$ *as well as functions* $\Lambda^s : T^p \to \mathbb{\Lambda}$ *and* $U^s : \mathbb{I}^s \to \mathbb{R}+$ *such that:*

1. *(Optimal Search) For all* $\tau = (\phi^s, G^s)_{s=a,b} \in T^p$ *and* $s \in a, b$,
   - *(a)* $U^s(i) \geq \tilde{U}^s(i, \tau, \Lambda^s(\tau))$ *for all* $i \in \mathbb{I}^s$;
   - *(b)* $\int_{\mathbb{I}^s} U^s(i) dG^s(i) = \int_{\mathbb{I}^s} \tilde{U}^s(i, \tau, \Lambda^s(\tau)) dG^s(i)$;
2. *(Impossible Terms-of-Trade) For all* $\tau = (\phi^s, G^s)_{s=a,b} \notin T^p$, *there is no* $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ *such that*
   - *(a)* $U^s(i) \geq \tilde{U}^s(i, \tau, \lambda^s)$ *for all* $s \in \{a, b\}$ *and* $i \in \mathbb{I}^s$; *and*
   - *(b)* $\int_{\mathbb{I}^s} U^s(i) dG^s(i) = \int_{\mathbb{I}^s} \tilde{U}^s(i, \tau, \lambda^s) dG^s(i)$ *for all* $s \in \{a, b\}$;
3. *(Profit Maximization)* $T = \arg\max_{\tau \in T^p} \tilde{V}(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$.

With these definition of partial equilibrium, the definition of a competitive search equilibrium with observable types is:

Definition 7 *A* ***competitive search equilibrium*** *is a partial equilibrium* $\{T^p, T, (\Lambda^s, U^s)_{s=a,b}\}$ *and a measure* μ *on the set of profit-maximizing terms-of-trade* T *such that:*

1. *(Free Entry)* $c = \tilde{V}(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$ *for all* $\tau \in T$;

2. *(Market Clearing)* $\forall s \in \{a,b\}$ *and* $I \subseteq \mathbb{I}^s$,

$$\int_I dF^s(i) I^s \geq \int_T \int_I N^s(\Lambda^a(\tau), \Lambda^b(\tau)) dG^s(i) d\mu(\tau)$$

with equality if $U^s(i) > 0$ *for all* $i \in I$.

3. *(Market Utility) For all* $s \in \{a,b\}$ *and* $i \in \mathbb{I}^s$ *with* $U^s(i) > 0$, $U^s(i) = \max_{\tau \in T} \tilde{U}^s(i, \tau, \Lambda^s(\tau))$.

We start by showing that the outcomes of a competitive search equilibrium can be characterized by a set of optimization problem. Consider the following problem given market utility U^s, $s = a, b$.

$$\tilde{V} = \max_{(\phi^s, \lambda^s, G^s)_{s=a,b}} m(\lambda^a, \lambda^b) \sum_{s=a,b} \int_{\mathbb{I}} \phi^s(i) dG^s(i), \tag{37}$$

$$\text{s.t. } U^s(i) \geq \lambda^s \left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j) - \phi^s(i) \right) dG^{\bar{s}}(j) \text{ with equality if } i \in supp(G^s),$$

$$\text{for } s = a, b \text{ and } i \in \mathbb{I}^s$$

Lemma 9 *Given* U^a, U^b, *a partial equilibrium* $\{T^p, T, (\Lambda^s, U^s)_{s=a,b}\}$ *can be constructed as follows:*

- T^p *is the set of all* $\tau = (\phi^s, G^s)_{s=a,b}$ *for which there exist* $(\lambda^s)_{s=a,b} \in \mathbb{\Lambda}$ *satisfying the constraints of Problem (37);*
- *For* $\tau \in T^p$, $\Lambda^s(\tau)$ *is the corresponding* λ^s *from the constraints of (37);*
- T *is the set of* $\tau \in T^p$ *such that* $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ *solves Problem (37).*

Conversely, for any partial equilibrium $\{T^p, T, (\Lambda^s, U^s)_{s=a,b}\}$ *and any* $\tau = (\phi^s, G^s)_{s=a,b} \in T$, *the tuple* $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ *solves problem (37).*

Proof. The proof of this lemma follows the identical construction steps as in the proof of lemma 1. ■

We first show that it is without loss of generality to focus on a competitive search equilibrium that is separating. It is convenient to define the joint surplus $f(i,j) \equiv u^a(i,j) + u^b(j,i)$. We define the problem of a separating market as

$$V^O(i,j) = \max_{\lambda^a, \lambda^b, \phi^a(i), \phi^b(j)} (1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma)^{1/\gamma} \left(\phi^a(i) + \phi^b(j)\right), \tag{38}$$

s.t.

$$U^a(i) = \lambda^a \left(u^a(i,j) - \phi^a(i)\right),$$

$$U^b(j) = \lambda^b \left(u^b(j,i) - \phi^b(j)\right).$$

It is without loss of generality to consider a separating equilibrium. This result comes from a non-separating market can be viewed as a collection of separating markets, with an additional restriction that all of these separating markets must share the identical contact rates. When $f(i,j)$ varies in the matching partners' types (dimension j), this identical contact rate constraint is binding and strictly decreases payoffs for the platform.

Lemma 10 *With observable types, either (1) all equilibrium terms-of-trade are separating, or (2) when there is an equilibrium with pooling terms-of-trade, there is another separating equilibrium such that agents have the same market utility, the same probability to match, the same expected fees paid, and the same partner distribution.*

Proof. We first replace $(\phi^a(i), \phi^b(j))$ in the objective function by the constraints in problem (37): The original problem is now an unconstrained problem in terms of (λ^a, λ^b) and (G^a, G^b):

$$\max_{(\lambda^s, G^s)_{s=a,b}} m(\lambda^a, \lambda^b) \int_{\mathbb{I}^a} \int_{\mathbb{I}^b} \left(f(i,j) - \frac{U^a(i)}{\lambda^a} - \frac{U^b(j)}{\lambda^b} \right) dG^b(j) dG^a(i).$$

Due to linearity, under the optimal solution, $m(\lambda^a, \lambda^b)\left(f(i,j) - \frac{U^a_i}{\lambda^a} - \frac{U^b_j}{\lambda^b}\right) = V^*$ for all $(i,j) \in supp(G^aG^b)$. Otherwise, a deviation that sets G^aG^b to be degenerate at the (i,j) with the highest integrand always improves the value of the objective function. Given any U^a and U^b, we can bound the value of the objective function:

$$V^* \leq V^O(i,j), \quad (i,j) \in supp(G^aG^b).$$

where the inequality comes from the fact $V^O(i,j)$ is a solution to problem (38) and any $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ is feasible for (38). It is useful to express both the pooling markets' problem and the separating markets' problem in terms of the agent-to-population ratios. More specifically, suppose there is a pooling market with an agent-to-platform ratio (n^a, n^b), fee $(\phi^a(i), \phi^b(j))$, and type distribution (G^a, G^b). and the separating market problem (38)(, after we eliminate the constraints) becomes:

$$V^O(i,j) = \max_{n^a, n^b} M(n^a, n^b) f(i,j) - n^a U^a(i) - n^b U^b(j).$$

Because M is strictly concave, the separating market problem has a unique solution. Denote this solution as $(n^{a*}(i,j), n^{b*}(i,j))$. Suppose we indeed have a pooling market in a competitive search equilibrium.

First, suppose $(n^{a*}(i,j), n^{b*}(i,j)) \neq (n^a, n^b)$ for some $(i,j) \in supp(G^a G^b)$. Because the separating market problem is strictly concave in (n^a, n^b):

$$V^* < V^O(i,j).$$

Now consider a deviation of a platform announcing: (a)$(\phi^{a*}(i,j), \phi^{b*}(i,j))$ for type (i,j) and infinity for all other types; (b) G^a degenerate at i and G^b degenerate at type G^b. This deviation delivers a higher payoff for the platform. A contradiction to the optimality of the original pooling terms-of-trade.

Second, suppose $(n^{a*}(i,j), n^{b*}(i,j)) = (n^a, n^b)$ for all $(i,j) \in supp(G^a G^b)$. In this case, consider an alternative equilibrium where, instead of having one pooling market with all the types, we have a collection of platforms designated for each pair of types. For the platform attracting type (i,j): (a) it announces $(\phi^a(i), \phi^b(j))$ for type (i,j) and infinity for all other types; (b) G^a degenerate at i and G^b degenerate at type G^b. Given the same market utility, matching probabilities will be exactly (λ^a, λ^b) and the agent-to-population ratios are (n^a, n^b); platforms break even. We set the measure of platforms designated for (i,j) to be $dG^a(i)dG^b(j)$. The collection of such platforms uses up the same measure of agents as in the original pooling market. Agents pay the same fees by construction when they find a match; In expectation across this collection of platforms, the match between (i,j) happens with the same probability. By doing this for all pooling terms-of-trade, we find a new competitive search equilibrium with separating terms-of-trades. Because the outcomes and payoffs are the same in every step, the new equilibrium is equivalent in terms of outcomes and payoffs in expectaion to the original equilibrium. ■

With separation, the problem (37) can be analyzed in two steps: (1) conditional on a fixed pair (i,j), find the $(\lambda^s, \phi^s)_{s=a,b}$ that maximizes the platform's value, as in problem (38). (2) choose (i,j) that delivers the highest value to platforms. We now derive the conditions for positive/negative sorting imposing the CES matching function.

Lemma 11 *Assume* $m(\lambda^a, \lambda^b) = (1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma)^{\frac{1}{\gamma}}$*:*

$$V^O(i,j) = \left[f(i,j)^{\frac{\gamma}{1+\gamma}} - (U^a(i))^{\frac{\gamma}{1+\gamma}} - (U^b(j))^{\frac{\gamma}{1+\gamma}} \right]^{\frac{1+\gamma}{\gamma}}.$$

Proof. We start by re-writing problem (38) by eliminating the fees by plugging in the constraints:

$$V^O(i,j) = \max_{\boldsymbol{\lambda}>0,\boldsymbol{\phi}>0} (1-(\lambda^a)^\gamma - (\lambda^b)^\gamma)^{1/\gamma} \left(f(i,j) - \frac{U^a(i)}{\lambda^a} - \frac{U^b(j)}{\lambda^b} \right)$$

It is more convenient to work with the matching function in terms of agent-advertisement ratios. With the CES matching function, we are solving equivalently

$$V^O(i,j) = \max_{n^a,n^b>0} (1+(n^a)^{-\gamma} + (n^b)^{-\gamma})^{-1/\gamma} f(i,j) - n^a U^a(i) - n^b U^b(j)$$

This is a strictly concave problem, thus the first-order conditions pin down the optimal solution:

$$\left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-\frac{1}{\gamma}-1} (n^a)^{-\gamma-1} = \frac{U^a(i)}{f(i,j)}$$

$$\left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-\frac{1}{\gamma}-1} (n^b)^{-\gamma-1} = \frac{U^b(j)}{f(i,j)}$$

Raising both side to the power of $\frac{\gamma}{1+\gamma}$:

$$\left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-1} (n^a)^{-\gamma} = \left(\frac{U^a(i)}{f(i,j)}\right)^{\frac{\gamma}{1+\gamma}}$$

$$\left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-1} (n^b)^{-\gamma} = \left(\frac{U^b(j)}{f(i,j)}\right)^{\frac{\gamma}{1+\gamma}}$$

Summing them:

$$1 - \left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-1} = \left(\frac{U^a(i)}{f(i,j)}\right)^{\frac{\gamma}{1+\gamma}} + \left(\frac{U^b(j)}{f(i,j)}\right)^{\frac{\gamma}{1+\gamma}}$$

and

$$(1+(n^a)^{-\gamma} + (n^b)^{-\gamma})^{-1/\gamma} = \left[1 - \left(\frac{U^a(i)}{f(i,j)}\right)^{\frac{\gamma}{1+\gamma}} - \left(\frac{U^b(j)}{f(i,j)}\right)^{\frac{\gamma}{1+\gamma}}\right]^{\frac{1+\gamma}{\gamma}}$$

Lastly:

$$V^O(i,j) = \left[f(i,j)^{\frac{\gamma}{1+\gamma}} - (U^a(i))^{\frac{\gamma}{1+\gamma}} - (U^b(j))^{\frac{\gamma}{1+\gamma}}\right]^{\frac{1+\gamma}{\gamma}}.$$

■

Given the results in Lemma 11, the step-two problem is to find the set of (i,j) that maximize platforms' value:

$$c = \max_{i,j} V^O(i,j). \tag{39}$$

In the next proposition, we consider the CES matching function and show our results resembles the ones in Eeckhout and Kircher (2010), although in our context the matching function takes three parties.

Proposition 7 (Sorting with Symmetric Information) *Assume* $m(\lambda^a, \lambda^b) = (1-(\lambda^a)^\gamma - (\lambda^b)^\gamma)^{\frac{1}{\gamma}}$*, if a competitive search equilibrium exists, it is a positively (negatively) sorted if* $f(i,j)^{\frac{\gamma}{1+\gamma}}$ *is super-modular (sub-modular).*

Proof. From lemma 11, we solve $\max_{i,j}\left[f(i,j)^{\frac{\gamma}{1+\gamma}} - (U^a(i))^{\frac{\gamma}{1+\gamma}} - (U^b(j))^{\frac{\gamma}{1+\gamma}}\right]^{\frac{1+\gamma}{\gamma}}$. If $f(i,j)^{\frac{\gamma}{1+\gamma}}$ is super-modular, then for any (i_1, j_1) and (i_2, j_2) such that $i_1 > i_2$ and $j_1 > j_2$

$$\begin{aligned}
&V^O(i_1,j_1)^{\frac{\gamma}{1+\gamma}} + V^O(i_2,j_2)^{\frac{\gamma}{1+\gamma}} - V^O(i_1,j_2)^{\frac{\gamma}{1+\gamma}} - V^O(i_2,j_1)^{\frac{\gamma}{1+\gamma}} \\
=&f(i_1,j_1)^{\frac{\gamma}{1+\gamma}} + f(i_2,j_2)^{\frac{\gamma}{1+\gamma}} - f(i_1,j_2)^{\frac{\gamma}{1+\gamma}} - f(i_2,j_1)^{\frac{\gamma}{1+\gamma}} \\
>&0
\end{aligned}$$

Suppose supermodularity and there is a competitive search equilibrium with non-positive sorting. WLOG, assume there are two terms of trades in this equilibrium attracting (i_1, j_2) and (i_2, j_1) with $i_1 > i_2$ and $j_1 > j_2$. Both terms-of-trades must satisfy the free-entry condition, so $V^O(i_1, j_1) = V^O(i_2, j_2) = c$, given the equilibrium market utility. We have:

$$V^O(i_1,j_1)^{\frac{\gamma}{1+\gamma}} + V^O(i_2,j_2)^{\frac{\gamma}{1+\gamma}} > V^O(i_1,j_2)^{\frac{\gamma}{1+\gamma}} + V^O(i_2,j_1)^{\frac{\gamma}{1+\gamma}} = 2c^{\frac{\gamma}{1+\gamma}}.$$

This means $V^O(i_1, j_2) > c$ or $V^O(i_2, j_1) > c$. A contradiction to the optimality condition. We can follow the same steps to establish the result for negative sorting. ■

Corollary 3 *Assume* $m(\lambda^a, \lambda^b) = (1-(\lambda^a)^\gamma - (\lambda^b)^\gamma)^{\frac{1}{\gamma}}$ *and* $u^s(i,j) = \left(\frac{1}{2}i^{\frac{\theta-1}{\theta}} + \frac{1}{2}j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$*, if a competitive search equilibrium exists, it is a positively (negatively) sorted if* $\theta < (>) 1+\gamma$.

Proof. In this case $f(i,j) = 2\left(\frac{1}{2}i^{\frac{\theta-1}{\theta}} + \frac{1}{2}j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$. Taking the cross-partials of $f(i,j)^{\frac{\gamma}{1+\gamma}}$, we have:

$$sign\left(f(i,j)^{\frac{\gamma}{1+\gamma}}\right) = sign(1+\gamma-\theta).$$

Thus, $f(i,j)^{\frac{\gamma}{1+\gamma}}$ is super(sub)-modular if $1+\gamma > (<\theta)$. ■