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Partially directed search in the labor market

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ABSTRACT

I study the labor market implications of an equilibrium search model with flexible degrees of information availability, which nests the random and directed search models as special cases. Workers have limited information about the payoffs of applying to different firms. Firms use wages to attract workers and mediate externalities among applicants. Limited information interacts with the allocative role of wages, leading to new predictions. Reducing information friction has non-monotonic impacts on efficiency. When the cost of acquiring information is low (high), alleviating the information friction reduces (exacerbates) the distortion in the market equilibrium. I then apply this model to discuss the implications of improvements in information technology and the spillover effects among workers in the labor market.

1. Introduction

Information is crucial for job search. Workers need to know the full set of relevant information to find their best matches, including the wage each job offers and the odds of being hired. However, assuming that workers have full access to this information is unrealistic. Only a small fraction of job postings contain explicit wage information (e.g., Marinescu and Wolthoff, 2020; Banfi and Villena-Roldán, 2019). Similarly, it is unrealistic to assume that workers do not have any information and search randomly for jobs. For example, high-wage vacancies attract more applicants (e.g., Belot et al., 2022).[2] The degree to which workers can direct their search has implications for our understanding of competition among employers for applicants and the allocation of workers across firms with different productivities. Equilibrium search theory assumes that workers have either full information (directed search) or no information (random search) about the relevant characteristics of jobs. Economists lack a tractable equilibrium framework to study the implications of partial information on wages and allocations.

This paper studies a theory of partially directed search with wage posting. I model limited information as a cost proportional to the Kullback-Leibler divergence (hereafter, *K-L divergence*) between the chosen search strategy and a random search strategy. One parameter governs how important this cost is, which I will refer to as the *cost of directing search*. When the cost of directing search tends to infinity, search is random. When the cost of directing search tends to zero, search is directed.

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[2] The elasticity between applicant-per-vacancy to wage is 0.7 to 0.9.

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This model of limited information is then embedded into an *entropic competitive search equilibrium*. This equilibrium concept generalizes the notion of a competitive search equilibrium into richer information settings. In the equilibrium, (i) heterogeneous firms maximize their profit, given the equilibrium market utility of workers and worker's optimal search decision; (ii) workers make their optimal search decision, given the equilibrium wages and job-finding probabilities at different firms; and (iii) the applicants to all firms add up to the exogenous measure of workers in the economy. The analytical tractability of the competitive search equilibrium also prevails in an environment with limited information.

At the theory's core is the interaction between limited information and the allocative role of wages. Endogenous wages allocate workers among different firms and mediate the *congestion externality* among applicants who search for the same job, where the congestion externality arises because workers do not internalize that their application decreases other workers' chances of getting a job. The availability of information, and consequentially how directed job search is, affects the allocative role of wages through two mechanisms. First, given a fixed set of posted wages, workers are more able to direct their search toward high-payoff jobs when the cost of directing search is low. I refer to this as the *direct effect*. Second, depending on the workers' ability to direct search, firms respond endogenously in their wage postings. I refer to this as the *wage-posting effect*. These two effects interact and have implications for worker-firm rent sharing and the allocation of workers among firms.

Costly directed search generates monopsony power even when firms are infinitesimal relative to the market. In equilibrium, the posted wage of a given job is the labor productivity, net of the congestion externality created on other applicants, and a markdown that is increasing in the cost of directing search. As the cost of directing search rises, workers are less able to target the better options in the labor market. The number of applicants becomes less elastic to wage changes. As a result, competition among employers softens, and the wage markdown widens. In one extreme, when search is random, firms capture all surplus from matches, as in the case of the Diamond paradox (Diamond, 1971). In the other extreme, when search is directed, workers are paid a share of the output, where the share equals their contribution in the matching process, as in a competitive search equilibrium.

The inefficiency of the decentralized equilibrium arises when the cost of directing search is intermediate. As the cost of directing search rises, the markdowns created by the lack of information widen. These markdowns create a wedge between the social value of applicants and the worker's payoff when some firms post wages equal to workers' outside options. Increasing the cost of directing search exacerbates this wedge through the wage posting effect. However, workers also become less responsive to the difference in firm-level payoffs as the cost rises. The direct effect implies that these wedges become less consequential for allocations. These countervailing effects imply that efficiency is non-monotonic in the cost of directing search. In the random search limit, both the efficient and the equilibrium outcomes are that workers apply to every firm with the same probability. In the directed search limit, the wedge vanishes. Thus, the efficient and the equilibrium allocation also coincide.

I apply the developed equilibrium framework to investigate the labor market implications of progress in information technology. When information frictions decline, workers reallocate from unproductive to productive firms. Due to the congestion in the job search process, the aggregate job-finding probability falls. Firms also endogenously respond in their wage postings. With a lower cost of directing search, unproductive firms face more competition and less congestion, and both forces push up their posted wages. Productive firms face more competition but also more congestion, and their posted wages can either increase or decrease. It is possible for both the aggregate job-finding probability and the average wage to fall with improved information technology.

I further extend the baseline model to study the spillover effects among heterogeneous workers in the labor market. I introduce heterogeneity on the worker side and endogenous job creation on the firm side. Some workers' skills are mismatched with the requirements of jobs, which creates a loss in productivity. To sort out the ideal workers, firms post high wages for workers with ideal skills and low wages for mismatched workers. More mismatched workers lead to a lower job-finding probability for everyone because firms are less likely to meet an ideal worker and are discouraged from creating jobs, creating a *composition externality*. This composition externality is maximized when search is random and is completely mediated when search is directed. The cost of directing search thus offers a flexible mechanism to alter the degree of the spillover effect.

This paper is organized as follows: I discuss the environment and characterization of both the finite game and the limiting equilibrium in Section 2, the efficiency of decentralized equilibrium and its policy implications in Section 3, and the application and extension of the baseline model in Section 4. All proofs are included in Appendix A.

Related literature. This paper is related to research in search theory, labor market power, and bounded rationality.

Search theory is based on the premise of a lack of information (Stigler, 1961). The search literature has developed primarily along two lines of research that assume either full or no information. Random search models assume searchers do not have information regarding whom to meet. Workers search, meet, and decide whether to match (Chade et al., 2017 summarizes this literature). Competitive search models since Montgomery (1991) and Moen (1997) have assumed perfect information. In a competitive search equilibrium, workers decide who to meet, search, and then decide whether to trade (Contributions to this literature are listed in the References; Wright et al. (2021) provides a comprehensive summary of the literature.) The intermediate case with partial information is less studied (Wright et al., 2021). My paper provides a tractable middle ground for these two classes of search models. It inherits the competitive nature of the directed search models, yet it features a flexible degree of randomness in matching.

This paper is closely related to research by Cheremukhin et al. (2020), who were the first to utilize the entropy cost in a matching model in which the payoffs are negotiated after a match is formed. Pilossoph (2014) and Lentz and Moen (2017) also apply the reduced-form Logit decision rule to models with Nash bargaining.

My paper uses a similar entropy cost as in Cheremukhin et al. (2020). The major difference in my model is the allocative role of wages. In their setting, wages are either fixed or determined by ex-post bargaining, and firms cannot use different wage postings to attract workers. In my model, wages are endogenously determined by the firms' tradeoff between more applicants and higher profit per worker. New positive and normative predictions arise when the wage posting effect is present. This wage-posting effect leads to non-monotonic implications of the efficiency impact of reducing information friction (I discuss in detail in section 3). This wage-posting effect also leads to new positive predictions of reducing information friction on unemployment and average wage (I discuss in detail in section 4).

There are several ways to consider flexible information in a search model. Burdett and Judd (1983), Burdett and Mortensen (1998), and Acemoglu and Shimer (2000) consider an environment in which a fraction of searchers can compare two offers; Lester (2011), Choi et al. (2018), and Bethune et al. (2019) consider an environment in which a fraction of searchers are informed and direct their search. Menzio (2007) provides a cheap talk theory of partially directed search. My paper considers endogenous information frictions that come from the rational inattention of workers. This flexible learning process leads to tractability, even when there are endogenous search frictions and firm heterogeneity, which are difficult to incorporate into the existing models.

My paper also contributes to the study of firm market power. The idea that lack of information leads to market power traces back to the Diamond Paradox (Diamond, 1971): if all consumers randomly search for deals, then any positive switching cost leads to monopolistic pricing. My paper nests the Diamond (1971) case as a special instance when the cost of directing search is high. The efficiency implications of this paper are related to the literature on firm markup and distortions. Dispersion of markdowns leads to misallocation across firms, an insight also present in Atkeson and Burstein (2008), Edmond et al. (2015), and Berger et al. (2022). This paper offers an alternative source of variable markups: through information frictions in the search process.

Finally, this paper builds on the literature that studies the decision implications of rational inattention (e.g., Caplin and Dean, 2015, Matéjka and McKay, 2012, Matějka and McKay, 2015, Ravid, 2020). I extend the study of rational inattention to a setting with search frictions, and provide a limiting equilibrium concept that captures the essential mechanism of bounded rationality but remains tractable.

2. Model

The baseline model is static. There are two groups of agents: workers and firms. Workers are homogeneous, indexed by i. Firms are indexed by j. I start with a finite population of workers and firms and then consider the case when their population grows to infinity.

Each firm has one vacant job to fill. When filled, the job at firm j produces output $z_j \in (b, \infty)$. All agents have linear utility. If firm j hires a worker at wage w, the firm will receive a payoff of $z_j - w$ and the hired worker will receive a payoff of w. Because of search frictions, there might be unmatched workers and unmatched firms at the same time. Workers who fail to find a match will receive their outside option of b. Firms that fail to find a match will receive their outside option of 0. With the assumption $z_j > b$, there are always gains from trade of matches at every firm.

2.1. Finite numbers of workers and firms

I consider a wage posting game played among I workers and J firms in order to understand the strategic source of the monopsony power and its link to limited information. The wage posting game unfolds in four stages:

1. Firms simultaneously announce their wages, given other firms' wages and the probability of hiring associated with different wage announcements.
2. Workers choose the probability of applying to firms, given the wage announcements and other workers' application strategies.
3. Workers and firms are matched in a frictional process. The number of workers who show up at each firm follows a binomial distribution due to a lack of coordination.
4. Workers who get a job offer decide whether to accept or reject it.

This game is identical to the one considered in Burdett et al. (2001), except for the cost of directing search. If workers apply to firms with probability $\mathbf{q} = (q_1, ..., q_J)$, then they need to pay a cost proportional to the K-L divergence of the chosen probability $\mathbf{q}$ from a uniform application probability $(\frac{1}{J}, ..., \frac{1}{J})$:

$$\text{Cost of Directing Search} = c \sum_{j=1}^{J} q_j \log \frac{q_j}{1/J}.$$

Specifically, the cost of directing search is a per-unit cost of search c multiplied by the expected likelihood ratio between $\{q_j\}$ and a uniform distribution $\{1/J\}$, evaluated using the distribution $\{q_j\}$.[3] I consider a symmetric subgame perfect equilibrium. The equilibrium is defined on the optimal application strategy of workers given a vector of posted wages $Q(\mathbf{w}) \equiv \left\{Q_j(\mathbf{w})\right\}_{j=1}^{J} \in \Delta^J$ and a vector of posted wages $\mathbf{w}^e \equiv \left\{w_j\right\}_{j=1}^{J} \in \mathbb{R}_+^J$, where Δ^J denotes the space of J-dimensional simplex.

[3] The qualitative results also hold for a general set of divergence measures called f-divergence, which I discuss in the Online Appendix.

Definition 1 *(Symmetric subgame perfect equilibrium).* A symmetric subgame perfect equilibrium is $Q(\mathbf{w}) : \mathbb{R}_+^J \to \Delta^J$ and $\mathbf{w}^e \in \mathbb{R}_+^J$ such that:

1. $Q(\mathbf{w})$ maximizes each worker's payoff, given $\mathbf{w}$ and other workers apply with $Q(\mathbf{w})$:

$$Q(\mathbf{w}) = \arg\max_{q\in\Delta^J} \sum_{j=1}^{J} \frac{1-(1-Q_j(\mathbf{w}))^I}{IQ_j(\mathbf{w})} q_j \max\{w_j - b, 0\} - c\sum_{j=1}^{J} q_j \log\frac{q_j}{1/J}. \tag{1}$$

2. w_j^e maximizes firm j's profit, given $Q(\mathbf{w})$ and other firms' wages $\mathbf{w}^e/\{w_j^e\}$[4]:

$$w_j^e = \arg\max_{w} \left(1-(1-Q_j(\tilde{\mathbf{w}}))^I\right)(z_j - w), \tag{2}$$

where

$$\tilde{w}_{j'} = \begin{cases} w_{j'}^e, & \text{if } j' \neq j \\ w & \text{if } j' = j \end{cases}.$$

In Definition 1, condition 1 imposes subgame equilibrium. Given any vector of posted wages $\mathbf{w}$ and other workers using strategy $Q(\mathbf{w})$, each worker chooses the application strategy q to maximize their payoff. A symmetric subgame equilibrium requires that the optimal strategy of each worker happens to be $Q(\mathbf{w})$. They rationally expect that the job finding probability at firm j is $\frac{1-(1-Q_j(\mathbf{w}))^I}{IQ_j(\mathbf{w})}$, the probability that firm j successfully hires divided by the expected number of workers applying to firm j. Condition 2 requires that the posted wage by firm j in the equilibrium maximizes its profit, the probability of hiring multiplied by the profit $z_j - w$, given the optimal application strategy of workers $Q(\mathbf{w})$ and its competitors' posted wages.

Micro-foundation of the cost of directing search. The game with observed wages and the cost of directing search is equivalent to an information-gathering game in which workers do not observe wages but can acquire information about wages. In the information-gathering game, firms draw productivity from a common distribution and optimally decide what wages to pay. Workers understand the game's structure but face uncertainty on each firm's productivity realization. They choose a conditional distribution of signals regarding wages by paying a cost proportional to the reduction of entropy from the prior distribution to the posterior distribution of wages.

Under the refinement proposed by Ravid (2020), equilibrium wages and allocations in the information-gathering game are equivalent to the outcomes of the posting game in the current model. The intuition behind this equivalence is that workers gather information to make application decisions. Although the choice of signals is flexible and high-dimensional, they are all equivalent to a simple signal directly recommending where workers should apply for jobs. So optimizing on the conditional distribution is as if the workers are directly optimizing where to search for jobs.[5] The details of the proof are provided in the Online Appendix C.

Existence. I show that a symmetric equilibrium always exists for any combination of (I, J) and productivity distribution. This existence result establishes a game-theoretic foundation for the limiting economy, which will be defined as a limit of the symmetric equilibrium when the number of agents grows to infinity.

Proposition 1 *(Existence of symmetric equilibrium). A symmetric subgame perfect equilibrium exists.*

2.2. Two-by-two case

Consider a simple case in which there are only two workers and two identical firms ($z_j = z$). Given any wage announcement (w_1, w_2), worker i chooses an application probability (q_1, q_2) to maximize the difference between expected income and the cost of directing search:

$$\max_{q_1+q_2=1} q_1\left(1 - q_1^{-i} + \frac{q_1^{-i}}{2}\right)\max\{w_1 - b, 0\} + q_2\left(1 - q_2^{-i} + \frac{q_2^{-i}}{2}\right)\max\{w_2 - b, 0\} \tag{3}$$
$$-c\left(q_1\log\frac{q_1}{1/2} + q_2\log\frac{q_2}{1/2}\right).$$

Worker i faces uncertainty about the firm to which worker $-i$ is applying. The probability of getting hired conditional on applying to firm 1 is $1 - q_1^{-i} + \frac{q_1^{-i}}{2}$, which is the sum of the probability that the other worker does not apply and that the other worker applies and the firm randomizes to hire the worker i. Similarly, the job-finding probability of applying to firm 2 can be calculated.

[4] I use $\mathbf{w}^e/\{w_j^e\}$ to denote the vector of wages after w_j^e is eliminated from $\mathbf{w}^e$.

[5] The common productivity distribution assumption leads to using a random search strategy as a benchmark in the cost of directing search.

Conditional on getting hired, worker i gets an offer with wage w_j, as promised. She can also choose to walk away from the job offer and get her outside option b. The offer-acceptance decision is simple: workers accept the job if $w_j > b$ and turn down the job offer if $w_j < b$. I assume workers accept job offers when they are indifferent.[6]

Taking the first-order condition and imposing the symmetry between two workers, we find the subgame equilibrium probability of applying to two firms, $(q, 1-q)$, given any combination of wage announcements $(w, w_{\_})$:

$$c \log \frac{q}{1-q} = \left(1 - q + \frac{q}{2}\right)(w-b)^{+} - \left(q + \frac{1-q}{2}\right)(w_{\_} - b)^{+}. \tag{4}$$

There is always a unique solution to Equation (4), which I denote as $Q(w; w_{\_})$. In the limit of $c \to 0$, workers will only apply to the firm with the highest expected payoff, in which case search is directed. In the limit of $c \to \infty$, the subgame equilibrium is $q = 1 - q = \frac{1}{2}$. Workers will not deviate from the random search strategy, in which case search is random. With $c \in (0, \infty)$, workers apply to a firm with a higher probability if the expected payoff of applying there is higher than the expected payoff of applying to its competitor. However, because of the convexity of the cost function, the solution to Equation (4) is always in the interval $(0, 1)$. As a result, search is partially directed for $c \in (0, \infty)$.

$Q(w; w_{\_})$ is the labor supply curve for a firm, given that its competitor posts wage $w_{\_}$. The slope of this labor supply curve is governed by the cost of directing search c. One property of this labor supply curve is crucial for our understanding of equilibrium: it jumps down to zero when $w_j < b$. Workers will never take a job offer that is worse than their outside option. Firms will never post an unacceptable wage because $w_j < b$ will result in zero hiring, and it will be strictly dominated by $w_j = b$. This leads to a constraint $w \geq b$. I refer to this constraint as the participation constraint of workers.

The symmetric subgame perfect equilibrium is characterized by the following fixed-point problem:

$$w^e = \arg\max_{w} \; [1 - (1-q)^2](z - w),$$

s.t.

$$q = Q(w; w^e), \;\; w \geq b.$$

Given that the competitor posts the equilibrium wage, w^e, the focal firm chooses its own wage to maximize the expected profit. The expected profit is the product of the probability that at least one worker applies $\left[1 - (1-q)^2\right]$ and the profit per hiring $z - w$. The firm is subject to the labor supply curve from the subgame equilibrium and to the workers' participation constraint.

Lemma 1 *(Two-by-two case with homogeneous firms). If $z_1 = z_2 = z$, the equilibrium wage is given by*

$$w_1^e = w_2^e = w^* = b + \max\left\{\frac{z-b}{2} - 2c, 0\right\}.$$

Because both firms post the same wage w^e, in the equilibrium, no cost is paid to direct search ($q_1^e = q_2^e = \frac{1}{2}$). However, the equilibrium wage is a decreasing function in the cost of directing search. Increasing the cost of directing search makes it more difficult for workers to compare alternatives, and it reduces the competition between firms.

Imperfect information leads to the monopsony power of firms. This monopsony power is related to the Diamond paradox: in a random search environment, the equilibrium wage is the workers' outside option, even if the search cost is slight. When the cost of directing search is positive but finite, the wage is above workers' outside option. Here, the Diamond paradox is circumvented because of the competition in the wage posting game. Suppose both firms post the worker's outside option in the equilibrium. Each firm would find it optimal to increase wages when the cost of directing search is small enough. This increase in wages would lead to more applicants and a higher probability of hiring.

In the homogeneous firm case, the equilibrium queue at each firm is always $\frac{1}{2}$, regardless of the cost of directing search. To study the implications of this monopsony power for allocation, it is necessary to consider the cases that involve heterogeneous firms. The comparison of queues and wages and the comparative static with respect to c is summarized as follows.

Lemma 2 *(Two-by-two case with heterogeneous firms). If $z_1 > z_2$, then*

- *$q_1^e \geq q_2^e$ and $w_1^e \geq w_2^e$, with strict inequality if $w_1^e > b$.*
- *$|q_1^e - q_2^e|$ is decreasing in c, strictly if $w_1^e > b$.*

The more productive firm posts a higher wage and attracts more workers, and the difference in the queue lengths shrinks when the cost of directing search increases. Analyzing the allocation when there are more than two firms is challenging because each firm's decision depends on a vector of strategies from its competitors. In addition, firms behave strategically in the finite economy. There is distortion due to oligopolistic competition (Galenianos et al., 2011). The inefficiency due to oligopolistic competition and the complexity of analyzing many firms will vanish when the economy is sufficiently large that firms' impact on the market outcome is negligible.

[6] This assumption, which makes the payoff function of firms continuous when the wage approaches b from the right, ensures the existence of an equilibrium.

From finite economy to limiting economy. I now consider a limiting economy in which the number of firms and workers grows to infinity. This limiting-economy model serves two purposes. First, it highlights the inefficiency introduced by information frictions because the strategic interactions vanish as firms become infinitesimal relative to the market. Secondly, as the strategic interactions vanish, the equilibrium's characterization becomes tractable, allowing me to discuss the positive and normative implications of the newly developed model. The details of this convergence result are discussed in the Online Appendix A.

2.3. Limiting economy: entropic competitive search

Consider a case with measure μ of workers and measure 1 of firms. Workers are indexed by $i \in [0, \mu]$ and firms are indexed by $j \in [0, 1]$.[7] I make two modifications to adapt the model to the limiting economy.

First, the matching process is characterized by $n(q)$, where q is the queue length at a firm and $n(q)$ is the probability that this firm will meet a worker, the job-filling probability. The probability that workers who apply to this firm will meet this firm is $m(q) = \frac{n(q)}{q}$, the job-finding probability. I make the following standard assumptions: (1) both $n(q)$ and $m(q)$ are differentiable; (2) the job-filling probability is increasing in the queue ($n'(q) > 0$) and the job-finding probability is decreasing in the queue ($m'(q) < 0$); (3) there is congestion: $n''(q) < 0$ and $m''(q) > 0$; and (4) the elasticity of job-filling probability with respect to the queue, $\epsilon(q) = \frac{n'(q)q}{n(q)}$, is weakly decreasing.[8,9]

Second, workers' strategies are adapted to continuous distributions. The workers' search strategy is a probability density function (hereafter, pdf) on the interval of $[0, 1]$. Define this pdf as a_j. The cost of directing search is the K-L divergence between the chosen search strategy a_j and the continuous uniform distribution on the interval $[0, 1]$[10]:

$$\text{Cost of Directing Search} = c \int_0^1 a_j \log a_j \, dj.$$

Subgame equilibrium. I first analyze the subgame given any wage profile w. The workers' problem is in Equation (5). To maximize their payoffs, workers take as given the wage profile $w : [0, 1] \mapsto [b, \max_j z_j]$ and choose the probability density function of applying to firm j:

$$a^e = \arg\max_a \int_0^1 m(q_j) \max\{w_j - b, 0\} a_j \, dj - c \int_0^1 a_j \log a_j \, dj, \tag{5}$$

s.t.

$$\int_0^1 a_j \, dj = 1.$$

Taking the first-order condition of the worker's search problem and imposing symmetry $q_j = \mu a_j$, we reach a limiting economy version of the labor supply curve:

$$m(q) \max\{w - b, 0\} - c \log \frac{q}{\mu} = V. \tag{6}$$

Workers equalize the net benefit of applying to every firm to a constant V. With K-L divergence, V is also the expected net payoff before workers send out applications, hereafter referred to as the market utility of workers.[11] The solution to Equation (6) is unique for every w, given a fixed market utility V. Define this solution as $Q(w; V)$. This is the labor supply curve in the limiting economy. I provide a visualization of this supply curve in Fig. 1. Because of the K-L divergence, the labor supply curve in the limiting economy is iso-elastic. When the expected payoff of applying to a firm increases by one percent, the queue length to this firm increases by $\frac{1}{c}$ percent. To mimic the subgame perfect equilibrium in the finite economies, I further require that Equation (6) holds for all $w \in [b, \max_j z_j]$, including for the off-equilibrium wages.

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[7] As a slight abuse of notation, I use subscripts to denote mapping from firm identity to outcomes to be consistent with the finite economy.

[8] This generalized matching process nests the matching process in the finite game as a special case. This special case is called the urn-ball matching process in the search literature. More specifically, when I workers apply to a firm with probability Q, the probability of this firm meeting a worker is $1 - (1 - Q)^I$, and the probability that a worker gets an offer from this firm is $\frac{1-(1-Q)^I}{IQ}$. As $I \to \infty$ holding $IQ = q$, these two probabilities converge to $n(q) = 1 - e^{-q}$ and $m(q) = \frac{1-e^{-q}}{q}$.

[9] The fourth assumption is made by many papers in the search literature. In my setting, it leads to a form of Marshall's second law in labor demand. When a firm has more applicants, its willingness to pay for workers does not increase. This assumption is satisfied in most of the matching functions used in the literature, such as the urn-ball function and the CES matching function.

[10] I make this restriction to maintain mathematical coherence. Economically, if I approximate the degenerate distribution as the limit of continuous distributions with shrinking supports, it is not a very restrictive assumption. Although I cannot directly define the K-L divergence between a discrete distribution and a continuous distribution, I can take the limit of the cost associated with the sequence of continuous distributions as the cost for the degenerate distribution. The K-L divergence asymptotes to infinity when the support shrinks. For this reason, I exclude degenerate distributions from the choice set of workers.

[11] To see this: Integrate the first-order condition in (6) with weight a_j. This results in $V = \int_0^1 m(q_j) \max\{w_j - b, 0\} a_j \, dj - c \int_0^1 a_j \log a_j \, dj$.

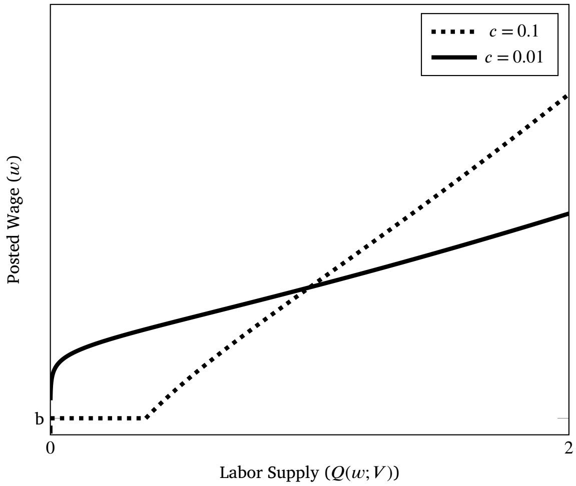

Fig. 1. $Q(w_j; V)$ - labor supply for different costs. *Note: The solid line plots the labor supply curve for the case with a high cost of directing search (c = 0.1). The dotted line plots the labor supply curve with a low cost of directing search (c = 0.01). Both supply curves take the market utility $V = 0.1$, matching function $m(q) = \frac{1-e^{-q}}{q}$, outside option $b = 1$, and $\mu = 1$. The labor supply curve is more elastic when the cost of directing share is low and is bounded below by the participation constraint $w \geq b$.*

Entropic competitive search equilibrium (ECSE). The equilibrium in the limiting economy inherits the spirit of competitive search models: firms maximize their payoffs, taking the wage-queue mapping in equilibrium as given. Yet this equilibrium differs from the standard competitive search equilibrium in its assumption about how much information is available to workers.[12] I refer to this new equilibrium concept as the *entropic competitive search equilibrium*. The entropic competitive search equilibrium is a tuple $\{w^e, q^e, V^e\}$, where w^e_j is the wage posted by firm j, q^e_j is the equilibrium queue at firm j, and V^e is the market utility of workers.

Definition 2 *(Entropic competitive search equilibrium)*. An entropic competitive search equilibrium is $\{w^e, q^e, V^e\}$ such that:

(i). (optimal posting) w^e_j solves firm j's profit maximization problem given $Q(w; V^e)$:

$$w^e_j = \arg\max_{w \in [b, z_j]} n\left(Q(w; V^e)\right)(z_j - w),$$

(ii). (optimal search) q^e is consistent with the subgame equilibrium given w^e

$$q^e_j = Q(w^e_j; V^e),$$

(iii). (market clearing) the total measure of queue equals the exogenous measure of workers:

$$\int_0^1 q^e_j \, dj = \mu,$$

For all $w \in [b, \max_j z_j]$, $Q(w; V^e)$ is the solution to the following equation:

$$m(q) \max\{w - b, 0\} - c \log \frac{q}{\mu} = V^e.$$

Characterization. Consider the firm j's problem given the equilibrium market utility V^e. Firm j faces a constrained optimization problem as in Equation (7):

$$\max_{w,q} \quad n(q)(z_j - w), \tag{7}$$

s.t.

$$m(q)(w - b) - c \log \frac{q}{\mu} = V^e, \quad w \geq b.$$

The profit of firm j is the probability of hiring $n(q)$ times the profit per worker $z_j - w$. The feasible combinations of (q, w) must be consistent with the labor supply curve $Q(w; V)$ and the participation constraint of workers.

[12] In a competitive search equilibrium, workers have full information.

Rewriting the problem only in terms of queue q_j helps the interpretation of the optimal decisions. Because of the property of the matching function $n(q) = qm(q)$ and the definition of the labor supply curve $Q(w; V^e)$, the firm's problem can be written as an unconstrained problem in q:

$$\max_q n(q)(z_j - b) - q\left(V^e + c\log\frac{q}{\mu}\right), \tag{8}$$

s.t.

$$V^e + c\log\frac{q}{\mu} \geq 0.$$

Similar to the competitive search equilibrium, the firms' problem in the entropic competitive search equilibrium could be reinterpreted as a problem of choosing the number of applicants to maximize its expected profit. To increase the queue, it must announce a higher wage. The total expected cost of recruiting for the firm is $q\left(V^e + c\log\frac{q}{\mu}\right)$, the quantity multiplied by the promised value per applicant.

Lemma 3 *(Optimal posting). The optimal solution for firm j is (w^e_j, q^e_j) such that:*

(Optimal Queue)

$$\max\left\{\underbrace{m(q^e_j)(z_j - b)}_{\text{Expected Hiring}} + \underbrace{m'(q^e_j)q^e_j(z_j - b)}_{\text{Congestion Externality}} - \underbrace{c}_{\text{Markdown}}, 0\right\} = \underbrace{V^e + c\log\frac{q^e_j}{\mu}}_{\text{Average Recruiting Cost}}, \tag{9}$$

(Optimal Wage Posting)

$$w^e_j = b + \max\left\{\epsilon(q^e_j)(z_j - b) - \frac{c}{m(q^e_j)}, 0\right\}. \tag{10}$$

In the optimal posting decision, firm j equalizes the marginal benefit and the marginal cost of an additional applicant. Attracting an additional applicant increases the chance of hiring, which creates the value of $m(q^e_j)(z_j - b)$. Meanwhile, this additional applicant also decreases the job-finding probability of other workers applying to the same job due to the congestion externality. Firms internalize this externality in their calculation of marginal benefit. Due to limited information, the firm also extracts a markdown from wages. Under the optimal posting, the firm extracts a markdown.

The wage formula can be derived from inverting the labor supply curve. The optimal wage posting is the contribution of workers in the matching process $\epsilon(q^e_j)(z_j - b)$ net of the markdown $\frac{c}{m(q^e_j)}$. The queue-wage relationship at the firm level is crucial for the results in later sections. The queue length affects the wage in two ways. Suppose the queue increases at a firm. The first effect is through job-filling elasticity $\epsilon(q)$. An increase in the queues decreases job-filling elasticity and leads to a lower wage. The second effect is through markdown. As queue length increases, the job-finding probability falls. The markdown in the job search process maps less into wages. Both effects imply firms with longer queues post lower wages.

When the firm is too unproductive compared to its competitors, it finds that the optimal wage to post falls below workers' outside options. Because this will lead to zero hiring, as in the finite economy, the firm becomes constrained by workers' participation constraints and posts outside option b instead. Given the market utility V^e, we can find the threshold below which firms become constrained:

$$\bar{z} = b + \frac{c}{n'\left(\mu e^{\frac{V^e}{c}}\right)}.$$

Corollary 1. *If $z_1 > z_2$, then $w_1 \geq w_2$, and $q_1 \geq q_2$, with strict inequality if $w_1 > b$.*

How do equilibrium queues and wages depend on the productivity of firms? Equation (9) implies that more productive firms post higher wages and attract longer queues. All firms face the same upward-sloping labor supply curve. For the more productive firms, an additional applicant is more valuable given the same queue. Therefore, the profit-maximizing queue for more productive firms must be higher than for less productive firms. Given the labor supply curve, more productive firms must promise workers higher levels of expected payoffs, which equals the product of the job-finding probability and the wage. Because the job-finding probability is lower at these firms, their wages must be higher. I conclude this section with the proposition for the uniqueness and existence of the ECSE.

Proposition 2 *(Existence and uniqueness of ECSE). There is a unique entropic competitive search equilibrium.*

3. Efficiency

This section discusses the efficiency property of the entropic competitive search equilibrium. The entropic competitive search equilibrium is efficient when the cost of directing search is low or infinite. For intermediate levels of costs, workers apply too much to low-productivity jobs. A decline in the cost of directing search can worsen the distortion in the decentralized equilibrium due to the interaction between the direct effect and the endogenous response of posted wages.

I make the following simplifying assumptions. Because the equilibrium allocation only depends on $z_j - b$, I will normalize $b = 0$ to simplify notations. For economies where b does not equal zero, one can relabel the productivity as $z - b$ and derive the same allocation.

Efficient allocation. The planner instructs workers how to apply to firms, equivalent to choosing the queue length at firm j, q_j:

$$\max_{q_j} \quad \mathbb{W} \equiv \int_0^1 n(q_j) z_j dj - c\mu \int_0^1 q_j \log q_j dj,$$

s.t.

$$\int_0^1 q_j dj = \mu.$$

Due to a lack of coordination, the planner must instruct all workers to use the same strategy.[13] The net output of the economy equals the sum of outputs from different firms minus the cost of directing search of all workers, which we define as $\mathbb{W}$. The constraint requires that the search strategy chosen by the social planner has to be an appropriately defined distribution.

The first-order condition on the queue length at firm j characterizes the unique optimal allocation. Denote the solution to this planner's solution as q_j^*. It must solve the equation system (11):

$$n'(q_j^*) z_j - c \log \frac{q_j^*}{\mu} = V^*, \tag{11}$$

$$\int_0^1 q_j^* dj = \mu.$$

When the marginal worker applies to firm j, the probability of a match for firm j increases by $n'(q_j^*)$. Directing search towards firm j incurs a marginal cost $c \log \frac{q_j^*}{\mu}$. The socially optimal allocation must equal the net benefit of applying to firm j to a constant social value V^*. Holding other primitives fixed, a different value of the cost of directing search leads to a different efficient allocation, which is the unique solution to the equation system (11). I denote the utility V^* associated with any cost c as $V^*(c)$.

Welfare theorem. Comparing the allocation of the entropic competitive search equilibrium to the allocation of the planner's solution, I reach, in Proposition 3, the welfare theorems of partially directed search. Because the cost of directing search is the focus of comparative statics, I state the welfare results in relation to c and keep all other primitives fixed.

Proposition 3. *The equilibrium is efficient if and only if $c \le \bar{c}$ or $c = \infty$, where $\bar{c}$ solves:*

$$\min_j z_j = b + \frac{\bar{c}}{n'\left(\mu e^{-\frac{V^*(\bar{c}) - \bar{c}}{\bar{c}}}\right)}. \tag{12}$$

Taking other primitives as fixed, the equilibrium is efficient if the cost of directing search is infinite or sufficiently low. The threshold $\bar{c}$ is the cost level such that the least productive firms become constrained when other firms post wages above workers' outside option. For the cases with $c > \bar{c}$, the participation constraint of workers forces unproductive firms to extract smaller markdowns because their unconstrained optimal wage is below workers' outside option. (If they could, they would want workers to pay for a match.) As a result, the markdown is unevenly distributed among firms. The dispersion of markdowns creates a wedge between the social value of applying to firm j and the equilibrium value of applying to firm j. The incentive to apply to different firms is distorted when the markdowns are unevenly distributed. Productive firms have bigger markdowns and attract fewer workers than is socially optimal; unproductive firms have smaller markdowns and attract more workers than is socially optimal.

The inefficiency of the decentralized equilibrium is non-monotonic due to the interaction between the direct effect and the wage-posting effect. The wage-posting effect creates the wedge between the social value and the equilibrium payoff. Yet, how much this

[13] For example, the planner cannot instruct half of the workers to apply to firms $[0, 0.5]$ and the other half to $[0.5, 1]$.

wedge leads to the misallocation of workers depends on the direct effect. As the cost of directing search rises, the dispersion in markdowns increases. However, this dispersion has fewer allocative consequences as the ability to direct search decreases.

The role of wage posting. Endogenous wage posting is crucial for the non-monotonicity of efficiency. Here, I benchmark my model with one where wages are bargained after hiring. With bargaining, the wage at firm j is a constant share βz_j, where $\beta \in [0,1]$ is the workers' bargaining power. A bargaining equilibrium solves the following equation system in terms of queues q^b and market utility V^b. This system is derived from the labor supply curve and the market-clearing condition:

$$m(q_j^b)\beta z_j - c\log\frac{q_j^b}{\mu} = V^b, \tag{13}$$

$$\int_0^1 q_j^b dj = \mu.$$

As the physical environment stays the same in both the ECSE and the wage-bargaining equilibrium, they should be compared to the same efficient allocation in (11). When search is random, both the wage-posting equilibrium and the wage-bargaining equilibrium are efficient because workers have to apply to every firm with the same probability. When cost is finite, the bargaining equilibrium is generically inefficient because the planner values applicants at the margin, but workers internalize the average job-finding probability. This inefficiency cannot be resolved by any constant bargaining power when firms are different in their productivity. One would expect the distortion to be strongest when search is directed. In my model, the equilibrium is efficient when search is directed.

Policy implications. In the case where the information markdown is distortionary, what type of policy remedy would restore efficiency? A minimum wage is often proposed as a remedy to monopsony power. In this model, a binding minimum wage exacerbates the distortion. Suppose there is a binding minimum wage $\underline{w} \in (\min_j w_j^e, \min_j z_j)$. Because $\underline{w} < \min_j z_j$, all firms are still making positive profits in equilibrium, so they stay active. The only difference between the case with a binding minimum wage and the baseline environment is that firms now face a tighter constraint on the wage:

$$\max_{w,q} \quad n(q)(z_j - w),$$

s.t.

$$m(q)w - c\log\frac{q}{\mu} = V^e, \quad w \geq \underline{w}.$$

In an equilibrium with a binding minimum wage, firms are divided by a threshold productivity $\bar{z}$. Firms with productivity below $\bar{z}$ are constrained by the minimum wage and have a fixed queue $\underline{q}$. Firms with productivity above $\bar{z}$ are unconstrained. When the minimum wage increases, the threshold productivity $\bar{z}$ increases. More firms become constrained, and all firms below the new threshold are forced to pay higher wages. As a result, workers have a higher market utility, leading unconstrained firms to post a higher wage through competition.

As a result, posted wages increase for every firm after a minimum wage hike. Workers reallocate from the productive firms to the unproductive firms because the unproductive firms now extract an even lower markdown than the productive firms. The effect on the net output of the economy can be written explicitly as:

$$\mathbb{W}'(\underline{w}) = -\int_0^1 \gamma_j \frac{d\log q_j^e}{d\underline{w}} dj < 0. \tag{14}$$

An increase in the binding minimum decreases the efficiency of the equilibrium allocation because it creates more markdown dispersion across firms. The proper policy remedy should aim to equalize markdowns. In the Appendix, I show in detail that a progressive profit tax can decentralize the efficient allocation.

Discussion of the efficiency result. In the baseline model, markdowns per se do not create inefficiency in the equilibrium; what matters is the distribution of markdowns across firms with different productivities. Two critical assumptions made in the baseline model lead to the efficiency result.

First, there is no endogenous job creation or labor market participation. These two assumptions imply that the level of the average markdown in the economy does not distort job creation or the choice of labor force participation. In a fully-fledged model where firms also make ex-ante investments to create vacancies (discussed in extensions), and workers can decide whether to participate in the labor market search, the level of markdowns will also lead to inefficiency either by encouraging firms to create too many vacancies or by discouraging workers from participating in the search.

Second, the iso-elastic property of the labor supply curve derived from K-L divergence means that the markdown in the expected utility space is constant across firms. Thus, without the binding participation constraint, the markdowns are equalized across firms, leading to efficiency. In the Appendix, I discuss a more general and flexible class of divergence measures that can easily be incorporated into the equilibrium framework. Under the general specification of divergence, the markdowns can differ across firms even without the binding participation constraint. In these cases, the equilibrium is inefficient. However, the economic intuitions are similar to the baseline case.

Empirical relevance. The key predictions from the model that can be brought to the data are (1) the queue-wage elasticity and (2) the comparative statics regarding reducing the cost of directing search.

The cost of directing search governs the responsiveness of the number of applicants to the posted wage of firms. Suppose a researcher can observe wages and the number of applicants simultaneously, such as in the studies with data from online job search boards. From the model, the predicted elasticity is:

$$\log q_j \approx \frac{1}{\frac{c}{m(q_j)w_j} + 1 - \epsilon} \log w_j. \tag{15}$$

Empirical studies of online job search behavior (e.g., Marinescu and Wolthoff, 2020) and experiments (e.g., Belot et al., 2022) find that the elasticity of the number of applicants to wages is 0.7 to 0.9, meaning that a one-percent increase in posted wage increases the queue by $0.7 \sim 0.9$ percent. Assuming a matching elasticity of 0.5, my model would imply a markdown $\frac{c}{m(q)w}$ of $0.61 \sim 0.92$. Suppose the cost of directing search is zero; equation (15) implies a matching elasticity of below zero, which violates the assumption regarding matching elasticity. This elasticity points to an in-between environment where the cost of directing search is positive but finite.

The implications of the efficiency result echo those found by Berger et al. (2022): understanding the direction of the reallocation of employment caused by policy changes is important to assess the effect of policies. Relative to the extreme cases of random and directed search, this model offers an intermediate model flexible enough to match the magnitude of reallocation effects.

Through the lens of this model, one would expect this type of distortion to be more severe in labor markets where the information regarding firms is opaque and wage negotiation is rare. For example, recent studies of minimum wages in Germany (Dustmann et al., 2021) find a significant reallocation effect after the imposition of the minimum wage. Admittedly, the current model needs to include endogenous job creation to fully account for the magnitude and direction of this reallocation effect.

This model's mechanism could also be investigated by examining regulations that govern pay transparency in a firm wage setting. For example, recent changes in law in Colorado, California, and New York require firms to include a pay range in their job postings. Viewed through the lens of this model, we can interpret that the cost of directing search is lowered by these policy changes. We should expect wages to increase and a reallocation of workers from low-wage firms to high-wage firms, especially among firms where post-match negotiation is rare.

4. Applications

In this section, I consider two applications of the developed equilibrium framework. The first application considers the impacts of reducing the cost of directing search on labor market outcomes, which could be interpreted as improving information technology or introducing wage transparency policies. The second application extends the baseline model to include horizontal worker heterogeneity, highlighting how different degrees of directness in job search imply varying spillovers among worker types.

4.1. Information technology and labor market

The past decades have witnessed a rapid improvement in information technology. These improvements affect how workers search for jobs and wage setting in the labor market. Several contributions to the literature posit that the improvements change the efficiency of the matching function (e.g., Martellini and Menzio, 2020); others posit that they increase the share of searchers who engage in directed search (e.g., Lester, 2011). This section considers the effect of a fall in the cost of directing search on employment and wages.

Suppose the cost of directing search falls. In the new equilibrium, firms change their posted wages, and workers change their application strategies. The following lemma describes these comparative statics. All the statements are made based on a small change in the cost of directing search, with other parameters fixed.

Lemma 4. *For every cost of directing search c, there is a threshold $\hat{z}(c)$, such that for a small decrease in the cost:*

1. A firm with $z < \hat{z}(c)$ attracts a shorter queue and posts a higher wage.
2. A firm with $z = \hat{z}(c)$ attracts the same queue and posts a higher wage.
3. A firm with $z > \hat{z}(c)$ attracts a longer queue and posts a higher or lower wage.

The fall in the cost creates direct and wage-posting effects. Holding the equilibrium wages fixed, it is easier for workers to seek out and apply to firms with higher wages. This direct effect reallocates workers from low-wage and unproductive firms to high-wage and productive firms.

Firms also endogenously respond to the fall in the cost of directing search. As it becomes easier for workers to seek high-payoff jobs, competition among firms increases. This drives up wages at every firm. The congestion externality changes differently depending on whether the focal firm is losing or gaining applicants. Unproductive firms lose applicants. From the assumption that the job-filling elasticity is decreasing in q, this force pushes up wages at unproductive firms further, strictly if the job-filling elasticity is strictly decreasing. Productive firms gain applicants. The same force from job-filling elasticity pushes down wages at productive firms. In net, the wage-posting effect implies an increase in wages posted by unproductive firms. The effect on wages posted by productive firms is ambiguous.

Through aggregation, improved information technology leads to a decline in the aggregate job-finding probability. The aggregate job-finding probability is defined as the total hirings across all firms divided by the measure of workers: $M = \frac{1}{\mu}\int n(q_j)dj$. The response of the aggregate job-finding probability to the change in the cost of directing search is given by:

Corollary 2. *When the cost of directing search falls, the aggregate job-finding probability falls.*

Workers reallocate from firms where queues are shorter to firms where queues are longer. This reallocation increases the dispersion of queues at the firm level. Because the job-filling probability, $n(q)$, is concave, a higher dispersion in queues implies a decrease in the aggregate job-finding probability. Interpreted from an aggregate matching efficiency perspective, the aggregate matching efficiency falls when the cost of directing search falls.

The effect on average wages is ambiguous. Workers reallocate from low-wage firms to high-wage firms. The wage-posting effect implies that the posted wages at high-wage firms can decrease. When the posted wages decrease enough, the average wage in the economy falls. This intuition is similar to the intuition posed by Lester (2011). In Lester (2011), the congestion effect occurs when firms shift from accommodating informed searchers to uninformed searchers. In my model, it comes from the reallocation of searchers from less congested to more congested markets.

The role of wage posting. In a wage-bargaining equilibrium, wages are independent of the available information in the economy. So, the wages at the firm level do not change when information technology is improved. Only the direct effect is present. Workers reallocate to high-productivity (high-wage) firms when it is less costly to direct search. The aggregate job-finding probability falls due to the same congestion force, but the wage unambiguously rises because now workers are more likely to apply to high-wage firms.

4.2. Horizontal worker heterogeneity

In this section, I provide an extension of the baseline model to a setting in which workers are heterogeneous in their skills. This extension highlights how the model in this paper provides a flexible framework for the discussion of the spillover effects among workers in wage determination and the job-finding probability.

There are two additions to the baseline model. First, firms have free entry to post a single job, which costs κ to create. Firms differ in their skill requirement $r \in \{1, ..., S\}$, which can be chosen freely upon entry. This requirement cannot be changed in the hiring process. Second, workers are heterogeneous and uniformly distributed in their skills $s \in \{1, ..., S\}$. When a match happens, productivity depends on whether workers have the *proper* skill for the requirement, whether or not $s = r$. If the skill is proper for the firm, the worker-firm pair produces z. Otherwise, the pair produces $z - x$ where $0 < x < z$. I set the population of workers to be 1 and the workers' outside option to be 0.

The number of skills S measures the potential skill mismatch in the economy. When S increases, the probability of a firm matching with the proper skill in a random draw decreases. I focus on comparative statics on wages and job-finding probability when S changes and how the comparative statics depend on the cost of directing search.

Skills are observable only after hires happen. This means different workers form the same queue and have the same job-finding probability at the firm level, but the wages can be posted type-specific. As in the baseline model, I focus on a symmetric equilibrium wherein workers search with the same strategy. The optimal search strategy for each worker follows a formula similar to the baseline model in equation (6) but is type-specific. For each $s = 1, ..., S$,

$$m\left(\sum_{s'=1}^{S} q_{s'}\right)\max\{w_s, 0\} - c\log\frac{q_s}{1/(S\nu)} = V_s, \tag{16}$$

where q_s is the queue length from type s workers, w_s is the posted wage for type s, ν is the endogenous measure of firms created, and V_s is the market utility of workers. There are three differences in the labor supply compared to the baseline model. First, the labor supply curve is type-specific because they can face different posted wages from the same firm and differ in market utility. Second, because types are not observable before the firm hires the worker, the sum of queues from all types, $\sum_{s'=1}^{S} q_{s'}$, determines the job-finding probability for every worker who applies to the same firm. Third, the costless strategy, $\frac{1}{S\nu}$, is endogenous because firms' entry decisions determine the measure of firms. For a given vector of wages $\mathbf{w} = \{w_S\}_{s=1}^{S}$, the equilibrium queues are the solution to the S equations in (16).

I leave the formal definition of equilibrium to Appendix A. The same steps as in the baseline model apply. Given the market utility of workers, firms choose wages to maximize their payoff. We look for market utilities that clear the market for workers. In principle, one must look for type-specific market utilities. In this stylized model, all workers face the same labor market conditions. In a symmetric equilibrium, they should have the same market utility. This reduces the number of market utilities to a scalar V^e.

From the firms' perspective, workers only differ in terms of whether their skills are proper. For an individual firm, I denote the wage and queue for workers with proper skill as (w_H, q_H) and the ones for other workers as (w_L, q_L). Following the same argument in the baseline model, I characterize the equilibrium by considering the firm's posting problem. More specifically, it is as if the firms directly choose the type-specific queue lengths:

$$\max_{q_H, q_L} m\left(q_H + (S-1)q_L\right)\left[z - (S-1)q_L x\right] - q_H\left(V^e + c\log\frac{q_H}{1/(Sv)}\right) - (S-1)q_L\left(V^e + c\log\frac{q_L}{1/(Sv)}\right). \tag{17}$$

The optimal choice of queues resembles the baseline model. The firms equalize the marginal benefit of increasing the queue from a type to its marginal cost. The following lemma summarizes the equilibrium:

Lemma 5. *In an ECSE with worker heterogeneity and entry, all firms post wage $\bar{w}_H$ to the skill-proper workers and $\bar{w}_L$ to the skill-improper workers, attracting a queue of $\bar{q}$. The wages and the queue are solutions to the following equations:*

(wage)

$$\bar{w}_H = \max\left\{z - (1-\epsilon(\bar{q}))\bar{z} - \frac{c}{m(\bar{q})}, 0\right\},\ \bar{w}_L = \max\left\{z - x - (1-\epsilon(\bar{q}))\bar{z} - \frac{c}{m(\bar{q})}, 0\right\},$$

and

(queue)

$$\kappa = n(\bar{q})(\bar{z} - \sigma\bar{w}_H - (1-\sigma)\bar{w}_L),$$

where

$$\sigma = \frac{S-1}{\exp\left[\frac{1}{c}m(\bar{q})(\bar{w}_H - \bar{w}_L)\right] + S - 1},\ \text{and}\ \bar{z} = z - (1-\sigma)x.$$

The wages follow a formula that is similar to the baseline model. In the equilibrium, all firms post the same set of wages and attract the same queue length. The equilibrium queue $\bar{q}$ is pinned down by the free-entry condition. Because the population is fixed at 1, the equilibrium queue is decreasing in the number of firms. Thus, a higher queue means firms create more jobs, and vice versa.

An increase in the number of skills, S, worsens the potential skill mismatch. Firms use wages to encourage skill-proper workers to apply while discouraging skill-improper workers. The ability to do so is affected by the cost of directing search. When search is directed, firms can always target the ideal workers. So, the expected productivity of a match is $\bar{z} = z$, and the loss due to skill mismatch is completely mediated. When search is random, the average productivity of a match is according to the population distribution, $\bar{z} = z - \frac{S-1}{S}x$.

The skill-improper workers create a negative spillover effect on other workers in terms of the job-finding probability. I start with the following lemma:

Lemma 6. *The equilibrium queue length $\bar{q}$ is increasing in S.*

As the potential skill mismatch worsens (S increases), the expected productivity of workers decreases. The firm responds by creating fewer jobs, and the job-finding probability decreases. As the job-finding probability decreases, the wage difference becomes less important, reinforcing the deterioration in expected productivity. In the Appendix, I show this semi-elasticity is always positive. The cost of directing search changes this elasticity. As the cost of directing search decreases, the elasticity also falls. In the limit of zero cost, this elasticity converges to 0.

The existence of skill-improper workers leads to an increase in wages for skill-proper workers. This happens through two forces. First, as expected productivity falls, the expected productivity for a match decreases. As a result, the congestion externality becomes less costly. This drives up the wages of skill-proper workers. Second, as firms respond by creating fewer jobs, the job-finding probability falls. Wages further increase because the job-filling elasticity $\epsilon(q)$ is decreasing in the queue length.

The cost of directing search provides one parameter that governs the spillover effects. To illustrate this, I will focus on the extremes. When the cost is infinity, all workers are paid their outside options and the free-entry condition requires $\kappa = n(\bar{q})(z - \frac{S-1}{S}x)$. The semi-elasticity of the queue to changes in S is $\frac{d\log\bar{q}}{dS} = \frac{1}{\epsilon(\bar{q})}\frac{x/S^2}{z-x+1/S}$. In this case, the skill mismatch is maximized, and firms fully respond to the change in the total surplus. When the cost is 0, it is always optimal in the equilibrium for the firms to single out the skill-proper workers, and $\frac{d\log\bar{q}}{dS} = 0$. Mismatch of skills has been argued in many papers in the literature as an important source of labor market friction. The model presented in this paper has the potential to provide a single parameter to summarize the interaction of heterogeneous workers in markets with search frictions.

5. Conclusion

This paper provides a tractable framework to study the equilibrium implications of limited information in the job search. The model nests random search models as a limit when the cost of directing search converges to infinity, and competitive search models are another limit when the cost of directing search converges to zero. I highlight that the interaction between the direct effect and the wage-posting effect is important to understand the positive and normative properties of the model. The developed model provides a flexible middle ground to model the spillover effects among workers in their labor market outcomes. The model in this paper is intentionally simple, and many realistic features of the labor market are assumed away. In the Online Appendix, I discuss possible extensions, which include vertical worker heterogeneity, dynamics, and a model in which the cost takes a general form other than the K-L divergence.

CRediT authorship contribution statement

Liangjie Wu: Writing – review & editing, Writing – original draft, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Omitted details and proofs

A.1. Proof of Proposition 1

This subsection proves the existence of a symmetric equilibrium in the wage posting game. I use the bold letter $\mathbf{w} = \{w_j\}_{j=1}^J$ and $\mathbf{Q} = \{Q_j\}_{j=1}^J$ to denote the wages and queues in the equilibrium. The strategy of proof is based on the results in Debreu (1952). I start by showing there is a subgame equilibrium of workers' search decisions given any wage announcement $\mathbf{w}$ and then show there exists a pure-strategy Nash equilibrium of the game of wage announcements among firms. The result from Debreu (1952) requires the action space of firms to be compact and convex, while the payoff function of firms is continuous and concave.

We start with the subgame given wage announcement $\mathbf{w}$. The last stage of the game is simple. Workers accept the job offer if the wage exceeds their outside option, $w > b$. In the game's third stage, workers are given wage announcements by firms and other workers' application decisions. They then choose which firm to apply to. Because we are looking for a symmetric equilibrium, I denote the strategy of other workers as $\{Q_j\}_{j=1,\dots,J}$. For each worker, the optimal search problem is:

$$\max_{q_j} \sum_{j=1}^{J} q_j \frac{1-(1-Q_j)^I}{IQ_j}(w_j - b)^+ - c\sum_{j=1}^{J} q_j \log\frac{q_j}{1/J},$$

s.t.

$$\sum_j q_j = 1.$$

The optimal decision of the worker is the solution to the first-order condition:

$$\frac{1-(1-Q_j)^I}{IQ_j}(w_j - b)^+ - c - \log q_j = \lambda,$$

where λ is the Lagrangian multiplier for the constraint. Imposing the constraint $\sum_j q_j = 1$ and imposing symmetry $q_j = Q_j$, we have:

$$Q_j = \frac{\exp\left(\frac{1}{c}\frac{1-(1-Q_j)^I}{IQ_j}(w_j - b)^+\right)}{\sum_{j'=1}^J \exp\left(\frac{1}{c}\frac{1-(1-Q_{j'})^I}{IQ_{j'}}(w_{j'} - b)^+\right)}.$$

Thus $\mathbf{Q} = (Q_1, \dots, Q_J)$ is a fixed point of a mapping $\mathbf{Q} = T(\mathbf{Q})$ where $T : [0,1]^I \to [0,1]^I$ is defined above. Because T is continuous and $[0,1]^I$ is a closed set, the Brouwer Fixed-point Theorem implies that a fixed point exists. So, a sub-game equilibrium given wage announcement $\mathbf{w}$ exists.

Firms will never post wages below workers' outside options, nor will they post wages above their own productivity. Thus, it is without loss generality to assume the queues are also bounded in $[\underline{Q}, \bar{Q}_j]$, where $\bar{Q}$ is the queue implied if the firm posts workers' outside options and $\bar{Q}_j$ is the queue if the firm posts z_j. For notational simplicity, I normalize $b = 0$. The cases with $b > 0$ can be accordingly derived. The subgame equilibrium requires that:

$$\frac{1-(1-Q_j)^I}{IQ_j}w_j - \log Q_j = V,$$

where V is the sum of all the constant terms that are not j-specific. For a focal firm j, its problem is.

$$\Pi(w;\mathbf{w}_{-j}) \equiv \max_{w\geq 0}(1-(1-Q(w;\mathbf{w}_{-j}))^I)(z_j - w).$$

Or equivalently, we can use the conditions to rewrite this problem in Q:

$$\Pi(Q;\mathbf{w}_{-j}) \equiv \max_{w\geq 0}(1-(1-Q)^I)z_j - IQ(V(Q;\mathbf{w}_{-j}) + c\log Q),$$

where $V(Q;\mathbf{w}_{-j})$ is the constant as defined in the subgame. No matter what wage the firm posts, it must be $\sum_{j'\neq j} Q_{j'} + Q = 1$.

For notational simplicity, I omit $\mathbf{w}_{-j}$ when it is unnecessary to include them. I want to show that $\Pi(Q)$ is a concave function. Taking the second-order derivative of this equation:

$$\Pi'(Q) = I(1-Q)^{I-1}z_j - I(V + c\log\frac{Q}{I}) - IQ(V'(Q) + \frac{c}{Q}),$$
$$\Pi''(Q) = -I(I-1)(1-Q)^{I-2}z_j - 2(IV'(Q) + \frac{c}{Q}) - \frac{c}{Q^2} - IQV''(Q).$$

If I can show $V'(Q) > 0$ and $V''(Q) > 0$, then $\Pi''(Q) < 0$. I prove this by totally differentiating the definition of $V(Q)$. Consider a change in Q. For firms with $j' \neq j$,

$$\xi_j \frac{dq_{j'}}{dQ} = \frac{dV}{dQ},$$

where $\xi_j = \frac{d}{dq}\left(\frac{1-(1-q_j)^I}{Iq_j}\right) < 0$. Add this condition across all $j' \neq j$:

$$\sum_{j'\neq j}\frac{dq_{j'}}{dQ} = \frac{dV}{dQ}\sum_{j'\neq j}\xi_{j'}^{-1}.$$

Because $Q + \sum_{j'\neq j} dq_{j'} = 1$, $\sum_{j'\neq j}\frac{dq_{j'}}{dQ} = -1$. Thus

$$\frac{dV}{dQ} = \frac{-1}{\sum_{j'\neq j}\xi_{j'}^{-1}} > 0.$$

Differentiate this expression:

$$\frac{d^2V}{dQ^2} = -\frac{\sum_{j'\neq j}\xi_{j'}^{-2}\frac{d\xi_{j'}}{dq_j}\frac{dq_j}{dQ}}{[\sum_{j'\neq j}\xi_{j'}^{-1}]^2}.$$

We can show $\frac{d\xi_j}{dq_j} > 0$ and $\frac{dq_{j'}}{dQ} < 0$ for $j' \neq j$. This implies that $\frac{d^2V}{dQ^2} > 0$. Together, these steps show that $\Pi''(Q) < 0$.

The choice set of firm j is a compact and convex set. The payoff function of firms is continuous in Q and concave in Q. From Debreu (1952), we know that a pure strategy Nash equilibrium in the first stage must exist. This concludes the proof.

A.2. Proof of Lemma 1

From the main text, the subgame equilibrium is summarized by the following equation. If the focal firm announces wage w, and its competitor announces wage w_-, they must be consistent with the subgame:

$$\left(1-q+\frac{q}{2}\right)(w-b) - \left(q + \frac{1-q}{2}\right)(w_- - b) = c\frac{q}{1-q}.$$

Thus, the optimal wage announcement decision of the focal firm is:

$$\max_{w,q}(1-(1-q)^2)(z-w),$$

s.t.

$$\left(1-q+\frac{q}{2}\right)(w-b) - \left(q + \frac{1-q}{2}\right)(w_- - b) = c\frac{q}{1-q}.$$

The constrained problem above can be written as an unconstrained problem in terms of q:

$$\max_q \left(1-(1-q)^2\right)(z-b) - 2q\left[c\log\frac{q}{1-q} + (q + \frac{1-q}{2})(w_- - b)\right].$$

Taking the first-order condition with respect to q, I reach the following first-order condition:

$$(1-q)(z-b)-(c\log\frac{q}{1-q}+(q+\frac{1-q}{2})(w^*-b))-q(c\frac{1}{q}+c\frac{1}{1-q}+\frac{1}{2}(w^*-b))=0.$$

In a symmetric equilibrium, firms post the same wage, and workers apply to one of them with the same probability. So I impose $q=\frac{1}{2}$ and $w=w^*$. This results in a wage $b+\frac{z-b}{2}-2c$. Imposing the participation constraint, I reach the result for the lemma:

$$w^*=b+\max\left\{\frac{z-b}{2}-2c,0\right\}.$$

Proof of Lemma 2

Start with the case where both firms post wages above b. In the proof of Lemma 1, 1 already derived the condition that determines the equilibrium queues. Recasting the same condition under productivity differential, I have the following conditions for the two firms:

$$q_2(z_1-b)-\left(c\log\frac{q_1}{q_2}+(q_1+\frac{q_2}{2})(w_2-b)\right)-q_1\left(c\frac{1}{q_1}+c\frac{1}{q_2}+\frac{1}{2}(w_2-b)\right)=0,$$

and

$$q_1(z_2-b)-\left(c\log\frac{q_2}{q_1}+(q_2+\frac{q_1}{2})(w_1-b)\right)-q_2\left(c\frac{1}{q_1}+c\frac{1}{q_2}+\frac{1}{2}(w_1-b)\right)=0.$$

Using the fact q_1+q_2 and plugging in the two conditions into the subgame equilibrium condition:

$$\left(1-q_1+\frac{q_1}{2}\right)(w_1-b)-\left(q_1+\frac{1-q_1}{2}\right)(w_2-b)=c\frac{q_1}{1-q_1}.$$

I can write the equilibrium as a non-linear equation in q_1

$$T(q_1)=0,$$

where

$$\begin{aligned}T(q)=&\frac{2-q}{3-2q}\left(q(z_2-b)-c\left(1+\frac{1-q}{q}+\log\frac{1-q}{q}\right)\right)\\&-\frac{1+q}{1+2q}\left((1-q)(z_1-b)-c\left(1+\frac{q}{1-q}+\log\frac{q}{1-q}\right)\right)\\&-c\log\frac{q}{1-q}.\end{aligned}$$

By taking the derivative, I can show $T'(q)>0$. Plugging in $q=\frac{1}{2}$, I have:

$$T(\frac{1}{2})=\frac{3}{8}(z_2-z_1)<0.$$

For there to be a crossing $T(q_1)=0$, it must be $q_1>\frac{1}{2}$. From the worker's optimal search decision, it must also be $w_1>w_2$. Additionally, I can show $\frac{\partial T}{\partial c}>0$. So an increase in c will move q_1^e closer to $\frac{1}{2}$, thus $q_1^e-q_2^e$ is decreasing in c.

Now consider the case when only one firm posts a wage $w>b$. In this case, the firm that posts wage above b solves the problem:

$$\max_w\,(1-(1-q)^2)(z_j-w),$$

s.t.

$$(1-q+\frac{q}{2})(w-b)=c\log\frac{q}{1-q}.$$

Writing in terms of q:

$$\max_q\,(1-(1-q)^2)(z_j-b)-2q(c\log\frac{q}{1-q}).$$

The first-order condition is

$$(1-q)(z-b)-c\log\frac{q}{1-q}-c(1+\frac{q}{1-q})=0.$$

If $w>b$,

$$\log\frac{q}{1-q}>0.$$

This implies the more productive firm (the one that posts a wage above b) attracts a longer queue than the unproductive firm (the one that posts wage b). The first-order condition also implies an increase in c leads to a decrease in the differences.

Proof of Lemma 3

Start from the firm's problem of choosing queues. I first ignore the participation constraint:

$$\max n(q)(z_j - b) - q\left(V^e + c\log\frac{q}{\mu}\right).$$

Taking the first-order condition, I reach:

$$n'(q)(z_j - b) - V^e - c\log\frac{q}{\mu} - c = 0.$$

If $n'(q)(z_j - b) - V^e - c > 0$, we have found the solution. Otherwise, the solution is $V^e + c\log\frac{q}{\mu} = 0$. Combining these two cases and using the fact $n'(q) = m(q) + m'(q)q$, we reach the condition in the lemma. The wage formula is an inversion of the labor supply curve.

A.3. Proof of Corollary 1

First, consider the case where both firms post wages above b. In this case:

$$n'(q_1)(z_1 - b) - c = V^e + c\log\frac{q_1}{\mu},$$
$$n'(q_2)(z_2 - b) - c = V^e + c\log\frac{q_2}{\mu}.$$

Taking the difference:

$$n'(q_1)z_1 - n'(q_2)z_2 = c\log\frac{q_1}{q_2}.$$

Suppose, to the contrary, $q_1 \le q_2$. The left-hand side of the equation is larger than 0 because $n(q)$ is concave and $z_1 > z_2$, while the right-hand side of the equation is positive. This is a contradiction. So it must be $q_1 > q_2$.

Second, consider the case where the firm with z_1 posts $w_1 = b$. I want to show $w_2 = b$ as well. Suppose, to the contrary, $w_2 > b$. This means $q_2 > q_1$. Because $n(q)$ is concave and $z_2 < z_1$, $n'(q_2)(z_2 - b) < n'(q_2)(z_1 - b)$. This is a contradiction. If both firms post b or $w_1^e > b = w_2^b$, the statement is trivially true.

A.4. Proof of Proposition 2

This section proves the existence and uniqueness of an entropic competitive search equilibrium. To show the existence, I rely on the continuity of the demand function. To show uniqueness, I rely on the strict monotonicity of the demand function. I start from the posting problem of firm j, taking as given any V^e:

$$\max_{w\ge b,q} n(q)[z_j - w],$$

s.t.

$$m(q)(w - b) - c[\log q - \log\mu] = V^e.$$

From the Maximum Theorem, the optimal solution q_j is continuous in V^e. The optimal queue at firm j solves the following first-order condition:

$$\max\{n'(q)z_j - c, 0\} - c(\log q - \log\mu) = V^e.$$

I want to show that the optimal queue length must be strictly decreasing in V^e. Suppose $n'(q_j)z_j > c$. Differentiating the first-order condition with respect to V^e implies

$$\frac{dq_j}{dV^e} = \frac{1}{n''(q_j)z_j - c\frac{1}{q_j}} < 0.$$

Suppose $n'(q_j)z_j < c$. Differentiating the first-order condition with respect to V^e implies

$$\frac{dq_j}{dV^e} = \frac{1}{-c\frac{1}{q_j}} < 0.$$

Because q_j is continuous in V^e, as V^e increases, q_j must strictly decrease for all $j \in [0,1]$. I now show there is a unique value of $V^e \in [0, \max_j z_j]$, such that $\int_0^1 q_j dj = \mu$. When $V^e = 0$, the constraint in the firm's optimization implies $q_j > \mu$. When $V^e = \max_j z_j$, the maximization implies that no firm posts a wage above its productivity. This means $w_j \le z_j \le \max_j z_j$. From the constraint:

$$c[\log q_j - \log \mu] = m(q) w_j^+ - \max_j z_j \le w_j^+ - \max_j z_j \le 0.$$

The first inequality uses $m(q) \le 1$. Thus, for any j,

$$q_j \le \mu.$$

A unique V^e exists such that $\int_0^1 q_j dj = \mu$. This concludes the proof.

A.5. Proof of Proposition 3

In this subsection, I prove the welfare theorem. I start by summarizing the planner's first-order condition:

$$n'(q_j^*) z_j - c \log \frac{q_j^*}{\mu} = V^*, \tag{18}$$

$$\int_0^1 q_j^* dj = \mu,$$

and the equilibrium condition:

$$\max\left\{ n'(q_j^e) z_j - c \right\} - c \log \frac{q_j^e}{\mu} = V^e, \tag{19}$$

$$\int_0^1 q_j^e dj = \mu.$$

If for any j, $n'(q_j^e) z_j > c$, then the $q_j^e = q_j^*$ are solutions to both equation systems. If there is some j such that $n'(q_j^e) z_j < c$, then $q_j^e \ne q_j^*$ for some j. The efficient allocation is unique because the planner's problem has a strictly concave objective function. Because the efficient allocation differs from the equilibrium allocation, the equilibrium allocation must be inefficient.

I have shown in the text that firms with higher productivity post higher wages and are further away from the participation constraint of workers, $w \ge b$. Thus, to find firms that are bound by the participation constraint, we need to focus on the firms with the lowest productivity. We define the threshold cost $\bar{c}$ such that:

$$\min_j z_j = b + \frac{\bar{c}}{n'\left(\mu e^{-\frac{V^*(\bar{c}) - \bar{c}}{c}}\right)}.$$

At this threshold of cost $\bar{c}$. If we solve the equilibrium queue of the firm with the lowest productivity, its queue length will be $\underline{q}$ such that $n'(\underline{q}) \min_j z_j = c$.

A.6. Proof of Lemma 4

From the characterization of the equilibrium:

$$\max\{n'(q_j^e)(z_j - b) - c, 0\} = V^e + c \log \frac{q_j^e}{\mu}.$$

Consider a small decrease in c. Differentiating the equation, I get:

$$n''(q_j)(z_j - b) dq_j^e - dc = dV^e + dc \log \frac{q_j^e}{\mu} + \frac{c}{q_j^e} dq_j^e.$$

Inverting this equation, I get:

$$\frac{dq_j^e}{dc} = \frac{1}{n''(q_j)(z_j - b) - c/q_j^e} \left(1 + \log \frac{q_j}{\mu} + \frac{dV^e}{dc} \right).$$

Due to concavity of $n(q)$, $n''(q_j)(z_j - b) - c/q_j^e < 0$ for all j. Thus, whether the queues increase or decrease depends on the sign of the terms in the parenthesis.

Given V^e, firms with different productivities have different queues, as shown in Corollary 1. Thus, the queue for some firms must change when the cost falls. Without loss of generality, I assume these firms increase their queues. With the market clearing condition, this also means the queue must fall for other firms. We also know q_j^e increases with productivity. Thus, if a firm increases its queue, all other more productive firms must also increase their queues. If a firm decreases its queue, all other less productive firms must also decrease their queues. This means there must exist a threshold $\hat{z}(c)$ such that the changes in queues follow the statement in the Lemma.

For the prediction of wages, we have derived the following wage formula:

$$w_j^e = \max\left\{\epsilon(q_j^e)z_j - \frac{c}{m(q_j^e)}, 0\right\}.$$

For a firm whose queue decreases, $m(q_j^e)$ increases, and $\frac{c}{m(q_j^e)}$ decreases. Because $\epsilon(q_j^e)$ is decreasing in q_j^e, all forces lead to an increase in the wage. The congestion externality implies that wages decrease at these firms, while the direct effect of a decline in cost implies an increase in wages. Thus, the net effect is ambiguous.

A.7. Proof of Corollary 2

From 4, the equilibrium queue profile with a lower cost of directing search is a mean-preserving spread of the equilibrium queue profile with a higher cost of directing search. By definition, $M = \frac{1}{\mu}\int_0^1 n(q_j^e)dj$. We differentiate this expression:

$$\begin{aligned}
\frac{dM}{dc} =& \frac{1}{\mu}\int_0^1 n'(q_j^e)\frac{dq_j^e}{dc}dj \\
=& \frac{1}{\mu}\int_{z_j>\hat{z}(c)} n'(q_j^e)\frac{dq_j^e}{dc}dj + \frac{1}{\mu}\int_{z_j\le\hat{z}(c)} n'(q_j^e)\frac{dq_j^e}{dc}dj \\
<& \frac{1}{\mu}\int_{z_j>\hat{z}(c)} n'(\hat{q})\frac{dq_j^e}{dc}dj + \frac{1}{\mu}\int_{z_j\le\hat{z}(c)} n'(q_j^e)\frac{dq_j^e}{dc}dj \\
<& \frac{1}{\mu}\int_{z_j>\hat{z}(c)} n'(\hat{q})\frac{dq_j^e}{dc}dj + \frac{1}{\mu}\int_{z_j\le\hat{z}(c)} n'(\hat{q})\frac{dq_j^e}{dc}dj \\
=& 0.
\end{aligned}$$

where $\hat{q}$ is the solution to:

$$\max\{n'(\hat{q})(\hat{z}(c) - b) - c, 0\} = V^e + c\log\frac{\hat{q}}{\mu}.$$

The first equality breaks the firms into two groups according to whether they gain or lose applicants. The first inequality uses the fact that, for firms that gain applicants, $\frac{dq_j^e}{dc} > 0$, and $n(q)$ is concave. The second inequality uses the fact that, for firms that lose applicants, $\frac{dq_j^e}{dc} < 0$, and $n(q)$ is concave. The last equality uses $\int_0^1 \frac{dq_j^e}{dc} = 0$.

A.8. Definition of ECSE with worker heterogeneity

Definition 3 *(ECSE with worker heterogeneity and free entry).* An entropic competitive search equilibrium with worker heterogeneity and free entry is $\{w^e, q^e, V^e, \nu\}$ such that:

(i). (optimal posting) $\{w_{sj}^e\}_{s=1}^S$ solves firm j's profit maximization problem given $Q_s(\{w_{sj}^e\}_{s=1}^S; V^e)$ and ν solves the optimal entry decision:

$$\kappa \ge \sum_{r=1}^S \frac{1}{S}\max_{\mathbf{w}} m\left(\sum_{s=1}^S Q_s(\mathbf{w}; V)\right)\sum_{s=1}^S Q_s(\mathbf{w}; V^e)\left(z - \mathbb{I}_{\{s\neq r\}}x - w_s\right),$$

with equality if $\nu > 0$.

(ii). (optimal search) q^e is consistent with the subgame equilibrium, given w^e

$$q_{s,j}^e = Q_s(\{w_{sj}^e\}_{s=1}^S; V^e).$$

(iii). (market clearing) the total measure of queues equals the exogenous measure of workers:

$$\int_0^\nu q_{s,j}^e dj = \frac{1}{S}.$$

Proof of Lemma 5

I will focus on the case with free entry. Start from the posting problem of firms and take the first-order condition:

$$\max\{m(q)z + m'(q)\bar{z} - c, 0\} = V^e + c\log\frac{q_H}{1/(Sv)},$$

and

$$\max\{m(q)(z-x) + m'(Q)\bar{z} - c, 0\} = V^e + c\log\frac{q_L}{1/(Sv)}.$$

Using the labor supply curve,

$$m(q^e)w_H = \max\{m(q^e)z + m'(q^e)\bar{z} - c, 0\},$$

and

$$m(q^e)w_L = \max\{m(q^e)(z-x) + m'(q^e)\bar{z} - c, 0\}.$$

Taking a difference between the labor supply curve:

$$c\log\frac{q_H}{q_L} = m(Q)(w_H - w_L).$$

Thus, the average productivity is given by:

$$\bar{z} = z - \frac{(S-1)q_L}{q_H + (S-1)q_L} = z - \frac{S-1}{\exp\left[\frac{1}{c}m(Q^e)(w_H - w_L)\right] + S - 1}.$$

Dividing through by $m(q^e)$ we get the wage formula as in the Lemma. Plugging these wages into the firm's profit and imposing the free entry condition, we get the equation that determines q^e.

Proof of Lemma 6

Start from the case in which both types are paid their outside options. In this case, $\bar{w}_H - \bar{w}_L = 0$, $\bar{z} = z - \frac{S-1}{S}x$, and firms take all the surplus. So, the equilibrium queue is given by:

$$\kappa = n(\bar{q})\left(z - \frac{S-1}{S}x\right) \implies n(\bar{q}) = \frac{\kappa}{z - x - \frac{1}{S}x}.$$

An increase in S increases the right-hand side of the equation. $n(q)$ is increasing. So, the queue length must increase in S. When both types are paid strictly above their outside options:

$$\kappa = \left(n(\bar{q}) - \bar{q}n'(\bar{q})\right)\left(z - \frac{S-1}{\exp\left[\frac{1}{c}m(\bar{q})x\right] + S - 1}x\right) + c\bar{q}.$$

Suppose S increases to $\tilde{S}$ and q decreases to $\tilde{q}$. Note that under $\tilde{q}$, the workers get paid higher wages and face higher job-finding probability. The market utility $\tilde{V}$ must be higher. Suppose firms choose a queue length of $\tilde{q}$ when the number of skill types is S and post wages according to the wage formula. This strategy leads to a lower wage than in the case with $\tilde{S}$ because $V < \tilde{V}$. This means:

$$\begin{aligned}\kappa =& n(\tilde{q})\left(z - \frac{\tilde{S}-1}{\exp\left[\frac{1}{c}m(\bar{q})x\right] + \tilde{S} - 1}x - \frac{\tilde{S}-1}{\exp\left[\frac{1}{c}m(\bar{q})x\right] + \tilde{S} - 1}\tilde{w}_L - \frac{\exp\left[\frac{1}{c}m(\bar{q})x\right]}{\exp\left[\frac{1}{c}m(\bar{q})x\right] + \tilde{S} - 1}\tilde{w}_H\right)\\ <& n(\tilde{q})\left(z - \frac{S-1}{\exp\left[\frac{1}{c}m(\bar{q})x\right] + S - 1}x - \frac{S-1}{\exp\left[\frac{1}{c}m(\bar{q})x\right] + S - 1}w_L - \frac{\exp\left[\frac{1}{c}m(\bar{q})x\right]}{\exp\left[\frac{1}{c}m(\bar{q})x\right] + S - 1}w_H\right),\end{aligned}$$

where the inequality comes from $\tilde{S} > S$. This means under S, a lower queue (more firms) than $\bar{q}$ yields a higher payoff to firms than the entry cost. This contradicts to $\bar{q}$ being the equilibrium queue length because more firms will find it optimal to enter under $\bar{q}$. A similar argument can be made regarding the case where the mismatched type is paid their outside options while skill-proper types are paid above their outside option.

A.9. Details for the minimum wage result

I first consider the impact of minimum wage on the market utility V. This market utility is increasing in the minimum wage. Notice the firms that are constrained by the minimum wage have a queue that solves the following equation:

$$m(\bar{q})\underline{w} - c\log\bar{q} = V.$$

If the market utility weakly decreases when the minimum wage increases, $\bar{q}$ will increase. For a weakly decreasing market utility, the queues at unconstrained firms weakly increase. As a result, the aggregate demand for applicants strictly increases. Given the old market utility clears the market and the equilibrium is unique, this is a contradiction.

For unconstrained firms, this first-order condition is

$$n'(q)z - c\log q - c = V.$$

I have already shown that market utility must rise when the minimum wage increases. The first-order condition implies that the queue for the unconstrained firms must decrease.

Next, I show the threshold of productivity must increase. Suppose, to the contrary, the threshold decreases. This means some firms that used to be bound by the minimum wage are now unconstrained, including the old threshold firms. Recall that threshold firms find minimum wage optimal. Our discussion so far implies that the queue at the old threshold firm must decrease. The new threshold firm is less productive than the old threshold firm and thus posts a lower queue than the old minimum wage queue. Therefore, fewer firms are posting the minimum wage, and the queue they demand from the market is decreasing. This cannot be true in equilibrium because the aggregate demand for applicants decreases for all firms.

Next, I turn to the results on wages. Posted wages increase for all firms. For firms that used to post the old minimum, their wages increased mechanically. The wages of unconstrained firms increase. To see this, notice the wage for an unconstrained firm with productivity z_j is

$$w_j = \max\left\{\epsilon(q_j)z_j - \frac{c}{m(q_j)}, 0\right\}.$$

I have shown q_j decreases for these firms. The matching elasticity $\epsilon(q_j)$ is decreasing, and $m(q_j)$ is decreasing in q_j. A decrease in q_j leads to an increase in wages.

Lastly, I show that the wages at the newly constrained firms also increase. To see this, notice that these firms used to be unconstrained and are now posting minimum wages. Suppose these firms are posting a wage higher than before. The new threshold firm used to be unconstrained and more productive than the newly constrained firms. This implies that the new threshold firm must post a wage higher than the new minimum wage, which is a contradiction.

A.10. Details for the profit tax result

A progressive profit tax can restore efficiency. Suppose the corporate profit tax is $T(\pi)$. This tax does not change workers' search decisions. Therefore, the firms in the economy still face the same labor supply curve. Posting a wage w generates after-tax profit $z_j - w - T(z_j - w)$ for firm j. Equation (20) summarizes the firm's problem with an arbitrary tax policy:

$$\max_{w,q} \; n(q)\Big(z_j - w - T(z_j - w)\Big), \tag{20}$$

s.t.

$$m(q)w - c\log q = V, \quad w \geq 0.$$

The goal is to design the shape of the tax function $T(\pi)$ that will decentralize the social planner's problem while guaranteeing that the workers are paid their social values. Proposition 4 states that there is a budget-balanced tax function that implements the planner's solution. Moreover, with this tax function, the equilibrium wage equals workers' contribution to the matching process. Therefore, the tax policy function also undoes the markdown due to the cost of directing search.

Proposition 4 *(Optimal corporate profit tax). The following budget-balanced tax function implements the social planner's solution:*

$$T'(\pi) = \frac{T(\pi) + \frac{c}{n'(q(\pi))}}{\pi + \frac{c}{n'(q(\pi))}},$$

where q^* is the solution to the social planner's problem and $\int_0^1 T(\pi_j^e)dj = 0$.

The optimal policy redistributes profits from the productive firms to the unproductive firms and workers. The productive firms are the ones that gain high profits in the equilibrium. The transfer policy increases the posted wage at all firms by making extracting markdowns less attractive. Unproductive firms are running lower profits due to the higher posted wage. Thus, the policy uses the tax revenue from productive firms to subsidize unproductive firms. This policy steers all firms away from the participation constraint $w \geq b$ and equalizes the markdown across all firms to 0.

A.11. Proof of Proposition 4

First, assume the tax function is well-behaved and maintains the objective function's strict concavity. Take the first-order condition given tax function $T(\pi)$,

$$n'(q)z_j - (V^e + c\log q) - c - n'(q)T(\pi) + \left[c - \frac{qm'(q)}{m(q)}(V^e + c\log q)\right]T'(\pi) = 0.$$

Comparing this equation to the planner's solution, I notice the wedge is

$$-c - n'(q)T(\pi)) + \left[c - \frac{qm'(q)}{m(q)}(V^e + c\log q)\right]T'(\pi).$$

The goal is to set a tax policy function such that the wedge is zero for every q, given the equilibrium market utility replicates the planner's solution V^*. According to the labor supply curve $V + c\log q = m(q)w$, the tax function needs to be such that

$$c + n'(q)T(\pi) = (c - qm'(q)w)T'(\pi).$$

With this wedge being zero, the wage must be

$$m(q)w = n'(q)z.$$

This implies

$$\pi = z - w = -\frac{qm'(q)}{n'(q)}w.$$

So, the tax function must be such that

$$\frac{c}{n'(q)} + T(\pi) = \left[\frac{c}{n'(q)} + \pi\right]T'(\pi).$$

The formula in the lemma can be derived by rewriting the equation above. To show the tax function is convex, I differentiate $T'(\pi)$:

$$T''(\pi) = \frac{-c\frac{n''q'}{(n')^2}(\pi - T)}{(\pi + \frac{c}{n'(q(\pi))})^2} > 0.$$

Online Appendix. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jet.2024.105858.

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