Assortative Matching with Private Information

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Abstract

We study matching between heterogeneous agents when their types are private information. Competing platforms post terms-of-trade. Agents with private information choose where to search and form matches. Positively assortative matching arises when high types are more likely to match with high types. We characterize an equilibrium with positively assortative matching when one exists, and provide sufficient conditions to ensure that matching is positively assortative in a limit with a vanishing cost for platforms. When more desirable partners have a higher willingness-to-pay for matches, they pay high fees to platforms to avoid less desirable types. When more desirable partners have a lower willingness-to-pay, they match at a low rate to keep out less desirable types.

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1 Introduction

Matching complementarities generate positively assortative matching, while substitutability generates negatively assortative matching (Koopmans and Beckmann, 1957; Becker, 1973). This conclusion holds in an environment where individual characteristics are common knowledge, so in particular everyone knows whom they are matching with when they match. This paper examines who matches with whom when individuals are privately informed about their own characteristics, and those characteristics affect the value an individual gets from matching and the value a partner gets from matching with them.

To have sorting in an environment where characteristics are completely hidden, there must be a mechanism for getting individuals with different types to separate themselves. The mechanism we explore is their choice of terms-of-trade by competing platforms, where a platform's terms-of-trade is defined as a combination of an access fee and a composition of other individuals in the platform. Individuals self-select into different platforms because they have different preferences over access fees and composition.

Importantly, we assume that there are three inputs into matching First, there are two types of agents who perform different roles in a match. They each search on a designated side of each platform. The platform may charge a different fee for accessing each side. As a result, the composition of individuals on each side of the platform may differ. Second, there is another kind of actor, homogeneous competitive platforms, which facilitate matching through costly postings to platforms. Free entry pins down the number of postings, and postings are only made at the most profitable terms-of-trade. When a platform matches a posting with a random, independently selected, and privately-informed individual from each side, the individuals pay the stated fees and then collect any benefit from matching.

This structure is rich enough to allow for both positive and negative sorting between the privately-informed individuals. Positive sorting arises if agents of higher types are more likely to match with partners of higher types. Negative sorting arises if higher-type agents are more likely to match with partners of lower types. Platform postings play an essential role in this analysis because they permit independent variation and ease of matching on both sides of the platform. For example, it may be easier to match on either side of one platform than on another. We find that this possibility is critical for positive sorting.

We motivate our model with a couple of examples. The first is a model of partnership, as in the labor or marriage market. Individuals are privately informed about their own characteristics, something that cannot be observed by platforms (e.g. job search or dating apps) or potential partners until after a partnership is formed. A critical object is the payoff that an individual with hidden characteristic i gets from matching with someone with hidden characteristic j, before paying any fee to the platform. We call this the payoff function $u_{i,j}$ and assume it captures everything that happens after i and j match and possibly learn each others' type. In the partnership model, it may be natural to assume that u increases in each of its arguments and is supermodular, so the cross-partial derivative is positive. The assumption that u increases in its first argument means higher types have a higher willingness-to-pay for any partner. The assumption that it increases in its second argument indicates that everyone agrees that higher types are more desirable partners. And supermodularity implies that higher types have a higher willingness-to-pay for partners.

In this example, we find that if there is positively assortative matching, higher types match at a higher rate, pay higher fees, and get higher utility from matching. In a limit where the cost of postings vanishes, separation occurs exclusively through differences in fees paid to the platform, with everyone matching at the same rate. In that limit, we also provide a sufficient condition for positively assortative matching.

A second example is one of disease transmission, such as HIV. Individuals are privately informed about their probability of being infected, again something that cannot be observed by platforms or potential partners, and prefer to match with healthier partners. More precisely, if the type is the probability of being healthy, then the value of a match increases in the partner's health status but decreases in one's own health status, since healthier people face a higher risk of becoming sick in any match. Finally, the payoff function is still supermodular, so a healthier person gains more from a given improvement in partner health than a sick person.

In this case, we find that if there is positively assortative matching, higher (healthier) types match at a lower rate, pay lower fees, and get less utility from matching, the opposite of the results in the partnership model. In a limit where the cost of postings vanishes, separation occurs exclusively through different matching rates, with all fees converging to zero. That is, healthy people distance themselves from the market by reducing the number of partners rather than trying to screen out undesirable partners through fees. Again, we provide a sufficient condition for positively assortative matching in this environment.

Our model is an extension of the competitive search equilibrium framework to an environment with two-sided private information. We build on Guerrieri, Shimer and Wright (2010) and Guerrieri and Shimer (2014), who analyzed competitive search equilibrium with one-sided private information. Our notion of competitive search equilibrium draws heavily on that earlier research. A key novelty in the present environment is that privately informed individuals care about who they match with, and their partner also has private information.

A terms-of-trade is defined by a pair of transfers from successfully matched agents on each side to the platform and a recommendation about which types of agents should show up on each side of the platform. Privately-informed agents direct their search to a particular side of a particular platform, and platforms make postings with the most profitable terms-of-trade. A constant return to scale matching function with these three inputs then delivers the number of matches between two agents and a platform. Through variation in the ratios of the three arguments of the matching function, we obtain independent variation in the ease of matching for the privately informed individuals on both sides of the platform. In equilibrium, the ease of matching on each side of a platform is determined endogenously, and platforms' recommendations about what types of individuals show up on each side of the platform must be fulfilled in any active platform.

In a competitive search equilibrium, platforms and privately-informed individuals have rational beliefs about the ease of matching on each side of each platform. They assume that their own behavior does not affect this. They then go to the platform and the side of the platform that delivers the highest expected utility. In turn, the recommendation about who is on each side of a platform and the belief about how hard it is to match on that side are consistent with individuals behaving rationally in their choice of platforms.

As is common in the competitive search literature, we prove that a competitive search equilibrium can be characterized through the solution to an optimization problem. However, unlike the existing literature, we find that without further assumptions, the equilibrium may have pooling, meaning that more than one type of individual comes to one side of active terms-of-trade. We obtain separation under two assumptions. First, we assume all types have a common ranking over partners, so the payoff function $u_{i,j}$ is increasing in j for all i. Second, we assume that higher types gain more from an increase in their partner's type, so the payoff function is supermodular in i and j. These assumptions also imply that incentive constraints can only bind downwards, so equilibrium is constrained only by the need to keep lower types out of the platform.

Prior research has explored sorting in competitive search equilibrium with observable types. Shi (2001) characterizes efficient sorting patterns and shows how they can be decentralized through a competitive search equilibrium. Heterogeneous firms post wages and skill requirements, and workers apply for the job, yielding the highest expected utility. Eeckhout and Kircher (2010) shows that negatively assortative matching can arise when there are matching complementarities, and find sufficient conditions for positively assortative matching in this environment.

Our main innovation relative to these papers is the introduction of private information, which means (in the language of Shi (2001)) that firms cannot post skill requirements but instead must accept the mix of workers who apply for the job. Additionally, we assume that both sides of the match care about their partner's type. In Shi (2001), workers only care about wages, not the type of firm they work for. In Eeckhout and Kircher (2010), sellers only care about the price they get, not who buys the object for sale.

There are also papers exploring sorting with information frictions. Hoppe, Moldovanu and Sela (2009) characterize assortative matching with private information. They assume that agents send costly signals and that matching is assortative in signals. Thus, there is positively assortative matching if higher types send higher signals. A single-crossing property ensures that this happens in equilibrium. Damiano and Li (2007) and Hoppe, Moldovanu and Ozdenoren (2011) characterize the screening of privately-informed types in a matching environment by a monopoly platform. Cai, Gautier and Wolthoff (2025) study sorting when workers are privately informed about their ability and heterogeneous firms screen them through interviews. We make three innovations relative to this literature. First, we allow for endogenous matching rates, so screening works both through platform fees and through contact rates. We show separation and screening can happen through either margin. Second, we consider a competitive search equilibrium, where platforms face competition from other platforms. This leads to different predictions regarding the separation of types into different markets. Lastly, we allow for more general payoff functions, showing that the characterization of equilibrium—whether separation works through platform fees or contact rates—depends critically on whether more desirable types have a higher or lower willingness-to-pay.

In Section 2, we develop a general framework for analyzing sorting in competitive search equilibrium when types are private information. Section 3 introduces some motivating economic examples to illustrate the usefulness of our framework. Section 4 analyzes equilibrium, including finding sufficient conditions for positively assortative matching. Section 5 focuses on the continuous-type case with symmetric agents on the two sides of the market, where we discuss sufficient conditions for positively assortative matching. Section 6 provides an algorithm to compute equilibrium when the two sides of the market are asymmetric or when there is negatively assortative matching. For comparison with the literature, we briefly discuss the necessity for endogenous posting by platforms and the conditions that lead to assortative matching when types are observable in Section 7.

2 Model

2.1 Platforms and Agents

We consider a static model of a two-sided market with three sets of risk-neutral participants: a-side agents, b-side agents, and platforms.

There exists a fixed measure I^s of s-side agents, $s \in \{a, b\}$. Each agent has private information about their type $i \in \mathbb{I}^s$, a compact subset of the unit interval. The distribution of types on side s is governed by an exogenous cumulative distribution function $F^s : \mathbb{I}^s \to [0,1]$.

When an s-side agent of type i matches with a type-j agent from the opposite side of the market, i receives a payoff $u^s(i,j)$ before paying any platform fees. We impose a key assumption on these preferences:

Assumption 1 (Common Ranking) *For every i, s and $j > j'$, $u^s(i,j) > u^s(i,j')$.*

This assumption requires that all agents on a given side agree on the ranking of potential partners from the other side. That is, if any type i prefers to match with j rather than j', then all other types i' share this preference. Given this commonality in preferences, we adopt the convention of labeling types such that a higher index indicates a partner who gives a higher payoff. Thus, $j > j'$ if and only if all agents prefer j to j'. Finally, we assume that u^s is continuous in its first argument.

There is also a fixed measure 1 of homogeneous platforms, each of which chooses its advertising effort at constant unit cost $c \geq 0$. This advertising effort facilitates matching between agents on the two sides of the market, with costs covered through endogenous fees paid by the agents.

2.2 Terms-of-trade and Payoffs

A terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b}$ specifies four objects: the fees ϕ^a and ϕ^b charged to matched agents on each side and the distributions G^a and G^b of agent types participating on each side. We allow either ϕ^a or ϕ^b to be negative but assume $\phi^a + \phi^b \geq 0$. We denote by $\mathbb{T}$ the set of feasible terms-of-trade, consisting of all tuples where ϕ^a and ϕ^b are real numbers with nonnegative sum, and G^a and G^b are probability distributions with support on subsets of $\mathbb{I}^a$ and $\mathbb{I}^b$, respectively.

Agents direct their search toward particular terms-of-trade τ. If an s-side type-i agent succeeds in trading at terms-of-trade $\tau \in \mathbb{T}$ with probability $\lambda^s \in [0,1]$, their payoff is

$$U^s(i,\tau,\lambda^s) \equiv \lambda^s \left(\int_{\mathbb{I}^{-s}} u^s(i,j)dG^{-s}(j) - \phi^s \right). \tag{1}$$

That is, with probability λ^s the agent trades, receiving an expected gross payoff equal to their expected utility $u^s(i,j)$ across potential trading partners drawn from distribution G^{-s}, less the platform fee ϕ^s.

Platforms facilitate these trades. For a platform advertising terms-of-trade $\tau \in \mathbb{T}$ with associated matching probabilities λ^s, the expected gross profit per unit of advertising effort is

$$V(\tau, \lambda^a, \lambda^b) \equiv m(\lambda^a, \lambda^b)(\phi^a + \phi^b) \tag{2}$$

where $m(\lambda^a, \lambda^b)$ denotes the platform's matching probability as a function of the agents' matching probability, discussed in the next subsection. Each unit of advertising effort at terms-of-trade τ results in matches with probability $m(\lambda^a, \lambda^b)$, generating fee revenue $\phi^a + \phi^b$. The platform's net profit subtracts the marginal advertising cost c from this gross profit.

2.3 Matching Function and Platform Matching Probability

To understand the platform matching probability m, we start with a more familiar object, the matching function M. We assume that in each market, the number of matches is a constant returns to scale function of the platform's advertising effort, the number of a-side agents, and the number of b side agents. Let α denote the advertising effort and $n^s \geq 0$ denote the number of s-side agents per unit of advertising effort at some terms-of-trade τ. Then the number of matches is $\alpha M(n^a, n^b)$, where $M : \mathbb{R}_+^2 \to [0, 1]$. Consistent with the constant returns to scale assumption, we assume that M is increasing, differentiable, and strictly concave in (n^a, n^b), with $M(n^a, n^b) \leq \min\{n^a, n^b\}$.

Agents' matching probability is related to the matching function via

$$\lambda^s = L^s(n^a, n^b) \equiv \frac{M(n^a, n^b)}{n^s} \leq 1. \tag{3}$$

for $s = a, b$ and $n^s > 0$. This is well-defined when n^a and n^b are both strictly positive and we handle the cases where n^a and/or n^b converge to either zero or infinity through some limits. The numerator is the matching probability per unit of advertising effort, while the denominator is s-side agents per unit of advertising effort, and so the ratio is matching probability per s-side agent. Using this, we can express the platform matching probability via the identity

$$m(L^a(n^a, n^b), L^b(n^a, n^b)) \equiv M(n^a, n^b).$$

This is the probability that the platform matches as a function of the agents' matching probabilities, rather than as a function of the agent-advertising ratios. It is this object that we use in equation (2).

By varying n^a and n^b, we trace out a set of feasible agent matching probabilities:

$$\mathbb{\Lambda} \equiv \{(L^a(n^a, n^b), L^b(n^a, n^b)) | n^a, n^b \geq 0\} \subset [0, 1].$$

The platform matching probability is a function $m : \mathbb{\Lambda} \to [0, 1]$. From the assumptions on the matching function, we get that $\mathbb{A}$ is a down set: if $(\lambda^a, \lambda^b) \in \mathbb{A}$, then $(\hat{\lambda}^a, \hat{\lambda}^b) \in \mathbb{A}$ for all $\hat{\lambda}^a \in [0, \lambda^a]$ and $\hat{\lambda}^b \in [0, \lambda^b]$.

We can also go back from the platform matching probability to the matching function by first finding the number of s-side agents per unit of advertising effort:

$$n^s = N^s(\lambda^a, \lambda^b) \equiv \frac{m(\lambda^a, \lambda^b)}{\lambda^s}, \tag{4}$$

and then recovering the matching function via the identity

$$M(N^a(\lambda^a, \lambda^b), N^b(\lambda^a, \lambda^b)) = m(\lambda^a, \lambda^b).$$

Parametric Example. A parametric example illustrates these properties and clarifies the domain restriction in $\mathbb{A}$. Suppose the number of matches per unit of advertising as a function of the number of s-side agents per unit of advertising follows a CES form:

$$M(n^a, n^b) = (1 + (n^a)^{-\gamma} + (n^b)^{-\gamma})^{-\frac{1}{\gamma}}, \tag{5}$$

where $\gamma > 0$. This matching function is continuous, increasing, strictly concave, and satisfies $M(n^a, n^b) \leq \min\{n^a, n^b\}$ for all $(n^a, n^b) \in \mathbb{R}^2_+$.

To express this in terms of matching probabilities, we first write equation (3) as

$$\lambda^s = L^s(n^a, n^b) = \left(\frac{1 + (n^a)^{-\gamma} + (n^b)^{-\gamma}}{(n^s)^{-\gamma}}\right)^{-\frac{1}{\gamma}}.$$

Inverting this gives us

$$n^s = N^s(\lambda^a, \lambda^b) = \left(\frac{1 - (\lambda^a)^{\gamma} - (\lambda^b)^{\gamma}}{(\lambda^s)^{\gamma}}\right)^{\frac{1}{\gamma}},$$

a version of equation (4). Substituting back into equation (5) gives

$$m(\lambda^a, \lambda^b) = (1 - (\lambda^a)^{\gamma} - (\lambda^b)^{\gamma})^{\frac{1}{\gamma}}. \tag{6}$$

We can equally go from the platform matching probability (6) back to the matching function (5).

By varying (n^a, n^b) across their support, we find that the feasible domain for matching probabilities is

$$\mathbb{A} = \{(\lambda^a, \lambda^b) | (\lambda^a, \lambda^b) \geq 0 \text{ and } (\lambda^a)^{\gamma} + (\lambda^b)^{\gamma} \leq 1\}.$$

This example satisfies all our assumptions: the matching probabilities have a bounded domain $\mathbb{\Lambda}$ that is down set, and the matching function M is increasing, differentiable, and strictly concave.[1]

2.4 Equilibrium

Our equilibrium concept builds on the competitive search literature pioneered by Moen (1997) and extended to environments with private information by Guerrieri, Shimer and Wright (2010). In these models, market-makers post terms-of-trade, and agents on both sides of the market direct their search toward their preferred terms. The resulting equilibrium combines price posting with rational expectations about matching probabilities.

The key innovation of the competitive search approach, relative to random search models, is that it allows market participants to use posted terms-of-trade to solve selection and incentive problems. As in Moen (1997), platforms in our setting act as market-makers who compete to attract agents. However, our setting differs in two important ways. First, following Guerrieri, Shimer and Wright (2010), we allow for private information about agent types. Second, platforms must simultaneously attract agents from both sides of the market, making this a two-sided matching problem with private information.

We break the definition of equilibrium into two parts. First, we define a partial equilibrium where everyone takes as given the agents' *market utility* $\bar{U}^s(i)$, the utility that a type-i agent on side s can obtain at their best possible terms-of-trade. The partial equilibrium concept requires that platforms cannot profitably deviate by offering a different terms-of-trade, taking these market utilities as given.

In the second step, we impose market clearing conditions that endogenously determine these market utilities. The market clearing conditions ensure that the total measure of each type choosing to trade equals the available supply of that type and that platforms earn zero profits.

More formally, we begin with the partial equilibrium definition:

Definition 1 *A* ***partial equilibrium*** $\{T^p, T, (\Lambda^s, \bar{U}^s)_{s=a,b}\}$ *is two sets* $T \subseteq T^p \subseteq \mathbb{T}$ *as well as functions* $\Lambda^s : T^p \to \mathbb{\Lambda}$ *and* $\bar{U}^s : \mathbb{I}^s \to \mathbb{R}+$ *such that:*

1. *(Optimal Search) For all* $\tau = (\phi^s, G^s)_{s=a,b} \in T^p$ *and* $s \in a, b$,

   (a) $\bar{U}^s(i) \geq U^s(i, \tau, \Lambda^s(\tau))$ *for all* $i \in \mathbb{I}^s$;

[1] We obtain points where $\lambda^s = 0 < \lambda^{-s}$ as by taking the limit as $n^s \to \infty$ for fixed n^{-s}. Points with $\lambda^a = \lambda^b = 0$ correspond to the limit as n^a and n^b both grow without bound. We get points with $(\lambda^a)^\gamma + (\lambda^b)^\gamma = 1$ by taking the limit as n^a and n^b converge to zero with a fixed ratio.

(b) $\int_{\mathbb{I}^s} \bar{U}^s(i)dG^s(i) = \int_{\mathbb{I}^s} U^s(i, \tau, \Lambda^s(\tau))dG^s(i)$;

2. *(Impossible Terms-of-Trade) For all* $\tau = (\phi^s, G^s)_{s=a,b} \notin T^p$, *there is no* $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ *such that*

   (a) $\bar{U}^s(i) \geq U^s(i, \tau, \lambda^s)$ *for all* $s \in \{a, b\}$ *and* $i \in \mathbb{I}^s$; *and*

   (b) $\int_{\mathbb{I}^s} \bar{U}^s(i)dG^s(i) = \int_{\mathbb{I}^s} U^s(i, \tau, \lambda^s)dG^s(i)$ *for all* $s \in \{a, b\}$;

3. *(Profit Maximization)* $T = \arg\max_{\tau \in T^p} V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$.

Let us first understand why the conditions in part 1 (Optimal Search) capture reasonable restrictions on behavior. Consider condition 1(a), which requires $\bar{U}^s(i) \geq U^s(i, \tau, \Lambda^s(\tau))$ for all agent types. Suppose this inequality were violated for some side s, type-i agent. Then these agents would strictly prefer this terms-of-trade to their market utility $\bar{U}^s(i)$, causing them to flock to this terms-of-trade. This influx would naturally drive down the matching probability $\Lambda^s(\tau)$ until the inequality is restored. Just as in traditional competitive search models, entry continues until agents are indifferent between this terms-of-trade and their market utility. Thus, any equilibrium must satisfy condition 1(a).

Condition 1(b) requires that agents who are supposed to come to the terms-of-trade (those in the support of G^s) obtain exactly their market utility on average. To see why this is necessary, suppose the average utility were strictly less than $\bar{U}^s(i)$ for types in the support of G^s. Then these agents would prefer to search elsewhere, making it impossible to sustain the promised type distribution G^s. Conversely, if the average utility were strictly greater than $\bar{U}^s(i)$, this would violate condition 1(a) for some types. Thus, 1(b) is required for the promised type distribution to be consistent with agents' directed search incentives.

The second component of the definition identifies terms-of-trade that are impossible given market utilities. For instance, a platform might post fees so high that no agent would be willing to pay them regardless of the matching probabilities. Alternatively, the promised type distribution G^s might be inconsistent with agents' incentives, for example, promising only high types will participate when in fact only low types would be willing to do so. The impossible terms-of-trade condition formalizes this by requiring that for any $\tau \notin T^p$, there exist no feasible matching probabilities, $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ that could make these terms-of-trade satisfy condition 1.

The third component determines which possible terms-of-trade are actually offered in equilibrium. Given the matching probabilities $\Lambda^s(\tau)$ for each possible terms-of-trade, platforms choose terms-of-trade $T \subseteq T^p$ that maximize their profit per unit of advertising. This optimization reflects platforms' optimal behavior given agents' responses encoded in the matching probabilities.

This definition extends the competitive search framework to accommodate two-sided matching with private information. The matching probabilities $\Lambda^s(\tau)$ play a role analogous to market tightness in traditional competitive search models, but now must simultaneously clear both sides of the market while respecting agents' incentives. The partition into possible and impossible terms-of-trade provides a tractable way to incorporate these incentive constraints into the equilibrium concept.

A partial equilibrium determines which terms-of-trade could emerge given fixed market utilities $\bar{U}^s(i)$. A competitive search equilibrium (CSE) endogenously determines these market utilities through free entry of platforms and market clearing.

Definition 2 *A **competitive search equilibrium** is a partial equilibrium $\{T^p, T, (\Lambda^s, \bar{U}^s)_{s=a,b}\}$ and a measure μ on the set of profit-maximizing terms-of-trade T such that:*

1. *(Free Entry) $c = V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$ for all $\tau \in T$;*

2. *(Market Clearing) $\forall s \in \{a, b\}$ and $I \subset \mathbb{I}^s$,*

$$\int_I dF^s(i) I^s \geq \int_T \int_I N^s(\Lambda^a(\tau), \Lambda^b(\tau)) dG^s(i) d\mu(\tau)$$

   with equality if $\bar{U}^s(i) > 0$ for all $i \in I$.

3. *(Market Utility) For all $s \in \{a, b\}$ and $i \in \mathbb{I}^s$ with $\bar{U}^s(i) > 0$, $\bar{U}^s(i) = \max_{\tau \in T} U^s(i, \tau, \Lambda^s(\tau))$.*

Let us examine each component of this definition. First, we introduce a measure μ over the set of profit-maximizing terms-of-trade T. This measure captures both the intensive margin (how much advertising there is at each terms-of-trade) and extensive margin (which terms-of-trade are offered). This formulation allows for a rich set of equilibrium outcomes, including ones where different platforms attract different types of agents at different terms-of-trade.

The free entry condition ensures that in equilibrium, no platform can earn positive profits net of the advertising cost c. If profits were positive at some terms-of-trade $\tau \in T$, more platforms would enter offering these terms until the profit is competed away. Conversely, if profits were negative, platforms would exit until profits rise to zero or that terms-of-trade disappears entirely.

The market clearing condition equates the supply and demand for each set of agents. For any measurable set of types I, the left side represents the total supply of these types (I^s is the total measure of s-side agents). The right side integrates across all active terms-of-trade to find the total measure of these types being used, accounting for the matching probabilities $\Lambda^s(\tau)$ and promised type distributions G^s.

The weak inequality in market clearing allows for the possibility that some agent types earn zero market utility and do not participate in any terms-of-trade. However, if all types in set I earn strictly positive market utility, then supply must exactly equal demand. This reflects that agents cannot be excluded from terms-of-trade offering them strictly positive utility in equilibrium.

Finally, market clearing does not impose any restrictions on sets with zero measure. That is, market clearing only needs to hold for almost every agent with $\bar{U}^s(i) > 0$. The last condition fills this gap, imposing that every agent who has positive market utility must get that level of utility in some terms-of-trade that gives platforms zero profits, i.e. at some $\tau \in T$.

2.5 Separation and Assortative Matching

We focus on two distinct characteristics of matching patterns that can emerge in equilibrium. The first concerns whether different types mix together on the same side of a single market, while the second concerns how types match across different markets. These patterns are important for characterizing equilibrium in two-sided markets.

First, we ask whether the equilibrium involves different types participating on the same side of a market, or whether types always separate into distinct markets. This motivates our definition of separation:

Definition 3 (Separation) *A terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b} \in \mathbb{T}$ is separating if for $s \in \{a, b\}$ there exists an i^s such that $dG^s(i^s) = 1$. A (partial or competitive search) equilibrium is separating if every profit-maximizing terms-of-trade $\tau \in T$ is separating.*

In a separating equilibrium, each active market attracts exactly one type from each side. This means that in equilibrium, an agent knows exactly who they will match with when they go to any active markets. This contrasts with pooling outcomes where multiple types might search in the same market.

Second, we examine whether there is systematic matching between agent types across markets. The classical notion of assortative matching describes whether high types match with other high types. In our setting with multiple markets, we need a definition that works across markets:

Definition 4 (Positive Assortative Matching) *Take any subset of terms-of-trade $\hat{T} \subset \mathbb{T}$. Select two elements $(\phi^s_k, G^s_k)_{s=a,b} \in \hat{T}$ for $k \in 1, 2$ and numbers i_k in the support of G^a_k and j_k in the support of G^b_k. If any such numbers satisfies $(i_1 - i_2)(j_1 - j_2) \geq 0$, then $\hat{T}$ has positively assortative matching (PAM). A (partial or competitive search) equilibrium has PAM if T has PAM.*

To understand this definition, consider two markets and pick any type i_1 that participates on the a-side of market 1 and any type i_2 that participates on the a-side of market 2. Similarly, pick types j_1 and j_2 from the b-sides of these markets. PAM requires that if $i_1 > i_2$, then $j_1 \geq j_2$. That is, if we find a higher a-side type in market 1 than in market 2, we must also find a weakly higher b-side type there. This captures the idea that higher types match together, but extends it to environments with multiple markets and possible pooling of types.

We can analogously define Negatively Assortative Matching (NAM) by requiring $(i_1 - i_2)(j_1 - j_2) \leq 0$. Under NAM, finding a higher a-side type in market 1 than in market 2 implies we must find a weakly lower b-side type there.

Importantly, separation and assortative matching are logically distinct properties. Consider a simple example with types $\mathbb{I}^s = \{0, 1\}$ and three markets in T:

- Market 1: a-side type 0 matches exclusively with b-side type 0
- Market 2: a-side type 1 matches exclusively with b-side type 1
- Market 3: a mixture of a-side types 0 and 1 both match with b-side type 0

This equilibrium exhibits PAM: whenever we find a higher a-side type in one market than another, we never find a strictly lower b-side type there. However, it is not separating because market 3 pools different a-side types. Conversely, markets could be separating but match high types with low types, violating PAM.

3 Motivating Examples

Before characterizing the equilibrium outcomes, we introduce three motivating examples.

Marriage Market. Suppose the agents are looking for partners for marriage. Here we interpret types as the attractiveness of agents, and their roles as men or women. The utility from forming a marriage depends on the private type of both the individual and their marriage partner. As a parametric example, we assume the utility function is:

$$u^s(i,j) = \left(i^{\frac{\theta-1}{\theta}} + j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}, \ \ \theta > 0.$$

In this example, $\theta > 0$ is the substitution elasticity. This utility function satisfies the common ranking assumption. Additionally, it is supermodular and satisfies increasing willingness-to-pay, two additional assumptions that we introduce in Section 4.

Labor Market. We can also use the same structure to obtain a model of the labor market. Spence (1973) assumed that workers' human capital was unobservable, while their education choices were observable. Here we assume that both firm productivity and worker human capital are unobservable and only transfers (i.e. wages) and queues (unemployment and vacancy rates) can be used to signal type. More precisely, assume side-a firms hire side-b workers. Then $u^a(i, j)$ is the output produced when a firm with productivity i hires a worker with human capital j and $u^b(i, j)$ is the nonpecuniary amenity that the worker derives from the match.

Market for Expertise. Suppose the side-a is the seller of an asset while side-b is the buyer. Sellers hold a single asset which may be of two types, A or B, with equal probability. Type A assets are naturally retained by the seller, while type B assets are more valuable if purchased by the buyer. For example, the asset may be a mortgage and the type may represent the difficulty of servicing the mortgage. The seller (mortgage issuer) may be better able to handle a hard-to-service loan, while the buyer (investment bank) may have the deep pockets to hold other loans. If the seller ultimately holds a type A asset, their payoff is 1, while it is -1 if they hold a type B asset. The buyer's payoffs are the opposite: 1 if they purchase a type B asset and -1 if they purchase a type A asset.

While the seller does not know the type of the asset, he knows how easy it is to observe the type, a number i. And while the buyer also cannot directly observe the type of the asset, he knows his expertise in assessing assets, a number j. If a type i seller matches with a type j buyer for an asset with quality A, they observe a normally distributed random variable with mean -1 and precision $\tau(i, j)$, an increasing function. If it is instead an asset with quality B, they observe a normally distributed random variable with mean 1 and precision $\tau(i, j)$. They then jointly decide whether to trade the asset and make any specified transfers.

It is straightforward to show that they will trade the asset if and only if the signal is positive. This means that the expected payoff of both the buyer and the seller from this meeting (before transfers) is

$$u(i, j) = \frac{1}{2} - \Phi(1, \tau(i, j)),$$

where $\Phi(1, \tau)$ is the probability that a normally distributed random variable with mean 1 and precision τ is negative. Again, these payoffs are increasing in both i and j and, under appropriate restrictions on τ, may be supermodular.

To understand the payoffs, proceed as follows: If the asset is of type A, with probability $1/2$, the trade occurs with probability $1 - \Phi(-1, \tau) = \Phi(1, \tau)$, resulting in payoff -1 for the buyer and 0 for the seller. If the asset if of type B, also with probability $1/2$, the trade occurs with probability $1 - \Phi(1, \tau)$, resulting in payoff 1 for the buyer and 0 for the seller.

Adding this up gives the buyer's expected payoff. The seller's expected payoff is constructed symmetrically.

Communicable Disease. Suppose individuals are looking to interact with each other, and their types are their probability of being healthy. If a healthy individual interacts with the sick individual, they may become sick, incurring an expected cost κ (the product of the probability of getting sick and the cost of being sick). Additionally, individuals gain 1 from an interaction. Sick individuals get 1 from interacting with either sick or healthy people.

This means that if a side-s agent who is healthy with probability i matches with an individual who is healthy with probability j, the value of the interaction is $u^s(i,j) = 1 - \kappa i(1-j)$. More specifically, the probability that the individual gets sick is proportional to the product of the probability that they are healthy i and that their partner is sick j. This payoff structure is similar to Philipson and Posner (1993) (for HIV/AIDS) and Farboodi, Jarosch and Shimer (2021) (for COVID-19). Interestingly, while the the payoff function is increasing in the partner's probability of being healthy and is supermodular, it is decreasing in the own probability of being healthy, since healthy people stand to lose more from interacting with sick people. Thus it satisfies decreasing willingness-to-pay, the flip side of increasing willingness-to-pay and another assumption we introduce in Section 4.

4 Characterization of Equilibrium

This section develops some preliminary theoretical results. We begin by establishing that market utilities must be continuous in equilibrium. We then show that finding a partial equilibrium is equivalent to solving a particular optimization problem. Next, we demonstrate that under supermodularity, all equilibria must be separating. Finally, we establish conditions under which market utilities are monotonic in agent types, allowing us to characterize equilibria through a system of differential equations.

4.1 Continuity of Market Utilities

Our first result establishes that market utilities must be continuous in any competitive search equilibrium. This property will be crucial for our subsequent analysis.

Lemma 1 *In any competitive search equilibrium, $\bar{U}^s$ is continuous for $s \in \{a, b\}$.*

The proof is in Appendix A.

This continuity property follows from agents' ability to mimic the strategies of nearby types. Intuitively, if market utilities had a discontinuous jump, types with a low market utility would enter the market for types with a high market utility, so such a market could not exist. This contradicts the market clearing condition.

4.2 The Platform's Problem

We next reformulate the partial equilibrium definition as a constrained optimization problem. This reformulation provides a tractable way to find equilibria and establish their properties. The key insight is that any terms-of-trade offered in equilibrium must maximize platform profits subject to constraints which ensure that the right type of agents are willing to come to the terms-of-trade and the wrong types are willing to stay away.

Consider a platform choosing terms-of-trade to maximize its profit per unit of advertising effort. Given market utilities $\bar{U}^s$, the platform solves:

$$\max_{\substack{(\lambda^a,\lambda^b)\in\mathbb{\Lambda},\\ \{\phi^s,G^s\}_{s=a,b}\in\mathbb{T}}} \quad m(\lambda^a,\lambda^b)(\phi^a+\phi^b) \tag{7}$$

$$\text{s.t.}\quad \bar{U}^s(i) \geq \lambda^s\left(\int_{\mathbb{I}^{-s}} u^s(i,j)dG^{-s}(j)-\phi^s\right)\ \forall i\in\mathbb{I}^s,\, s\in\{a,b\},$$

$$\int_{\mathbb{I}^s}\bar{U}^s(i)dG^s(i) = \int_{\mathbb{I}^s}\lambda^s\left(\int_{\mathbb{I}^{-s}} u^s(i,j)dG^{-s}(j)-\phi^s\right)dG^s(i),\ s\in\{a,b\}.$$

The objective function captures the platform's revenue from fees charged to both sides of the market, weighted by the probability of successfully forming a match. The first constraint ensures that no agent type can obtain utility higher than their market utility by going to this terms-of-trade. The second constraint requires that agents who do participate (those in the support of G^s) receive exactly their market utility.

Our first result establishes the close link between a solution to this optimization problem and a partial equilibrium:

Lemma 2 *Given $\bar{U}^a,\bar{U}^b$, a partial equilibrium $\{T^p,T,(\Lambda^s,\bar{U}^s)_{s=a,b}\}$ can be constructed as follows:*

- *T^p is the set of all $\tau=(\phi^s,G^s)_{s=a,b}$ for which there exist $(\lambda^s)_{s=a,b}\in\mathbb{\Lambda}$ satisfying the constraints of Problem (7);*
- *For $\tau\in T^p$, $\Lambda^s(\tau)$ is the corresponding λ^s from the constraints of (7);*
- *T is the set of $\tau\in T^p$ such that $(\Lambda^s(\tau),\phi^s,G^s)_{s=a,b}$ solves Problem (7).*

Conversely, for any partial equilibrium $\{T^p,T,(\Lambda^s,\bar{U}^s)_{s=a,b}\}$ and any $\tau=(\phi^s,G^s)_{s=a,b}\in T$, the tuple $(\Lambda^s(\tau),\phi^s,G^s)_{s=a,b}$ solves problem (7).

The proof is in Appendix A.

While it is always useful to restructure an equilibrium as an optimization problem, problem (7) remains challenging to solve directly. The difficulty stems from its mathematical structure: it is a mathematical program with equilibrium constraints, which exhibits inherent nonconvexity because increasing $dG^s(i)$ from zero changes inequality constraints into equality constraints. To make progress, we impose additional structure on preferences.

4.3 Supermodularity and Separation

In this section, we prove that if preferences are supermodular, equilibrium matching patterns exhibit a particularly simple structure. First, we state the supermodularity assumption:

Assumption 2 (Supermodularity) *For every $s \in \{a, b\}$, $i, i' \in \mathbb{I}^s$ and $j, j' \in \mathbb{I}^{-s}$ with $i > i'$ and $j > j'$, $u^s(i, j) + u^s(i', j') > u^s(i, j') + u^s(i', j)$.*

Combined with our earlier Common Ranking assumption, Supermodularity implies that higher types have a stronger preference for matching with higher partner types. This additional structure allows us to relate the competitive search equilibrium to a simpler optimization problem:

$$
\begin{aligned}
\max_{(\lambda^a,\lambda^b)\in\Lambda,\{\phi^s,k^s\}_{s=a,b}} \quad & m(\lambda^a, \lambda^b)(\phi^a + \phi^b) \\
\text{s.t.} \quad & \bar{U}^s(i) \geq \lambda^s(u^s(i, k^{-s}) - \phi^s) \ \forall i < k^s, i \in \mathbb{I}^s, s \in \{a, b\} \\
& \bar{U}^s(k^s) = \lambda^s(u^s(k^s, k^{-s}) - \phi^s) \ \forall s \in \{a, b\}, \\
& \phi^a + \phi^b \geq 0.
\end{aligned} \tag{8}
$$

Problem (8) differs from the problem (7) in two important ways. First, it considers only separating terms-of-trade in which exactly one type from each side participates in each market. Second, it includes only incentive constraints that prevent agents $i < k^s$ from deviating to terms-of-trade intended for higher types k^s. We ignore the constraints that prevent deviations to terms-of-trade intended for lower types.

Lemma 3 *Assume Common Ranking and Supermodularity. A tuple $\{T^p, T, (\Lambda^s, \bar{U}^s)_{s=a,b}\}$ together with a measure μ on T constitutes a competitive search equilibrium if and only if:*

1. *For each $\tau = (\phi^a, \phi^b, G^a, G^b) \in T$, there exist types $(k^a, k^b) \in \mathbb{I}^a \times \mathbb{I}^b$ such that the associated distributions are degenerate: $dG^a(k^a) = dG^b(k^b) = 1$;*

2. *For each $\tau = (\phi^a, \phi^b, G^a, G^b) \in T$, the fees ϕ^s and matching probabilities $\lambda^s = \Lambda^s(\tau)$ solve problem (8) given the types (k^a, k^b) and market utilities $\bar{U}^s$, with maximized value equal to c;*

3. *The set of possible terms-of-trade T^p and matching probabilities Λ^a, Λ^b satisfy the optimal search and impossible terms-of-trade conditions from Definition 1;*

4. *The measure μ satisfies the market clearing condition and $\bar{U}^s$ satisfies the market utility condition from Definition 2.*

The proof is in Appendix A.

The first two parts of Lemma 3 characterize the profit-maximizing terms-of-trade in a competitive search equilibrium under Supermodularity. First, such terms-of-trade attract one type from each side of the market, rather than a mixture over types. Second, such terms-of-trade solve problem (8), which in particular includes the constraints for keeping out lower types but drops the constraints from keeping out higher types. The last two parts of the Lemma tell us how to find the other elements of a competitive search equilibrium.

This Lemma yields two important implications for equilibrium structure. First, they immediately establish our main separation result:

Corollary 1 *Assume Common Ranking and Supermodularity. Any competitive search equilibrium is separating.*

To understand why pooling cannot occur in equilibrium, consider a market where multiple types participate on at least one side. Common Ranking implies that all participants would prefer to match with only the highest types from the other side of the market. This suggests creating a new market charging higher fees to attract only these high types. Supermodularity guarantees that lower types would be unwilling to pay the fees that make higher types indifferent, because the higher types value the improved matching enough to pay more than lower types could afford. While such a market might attract even higher types than those in the original pooling market, potentially making it infeasible, this just creates further profit opportunities. Even higher fees could be charged to attract those higher types. This process continues until complete separation is achieved.

An example illustrates the need for Supermodularity. Consider a market with $\mathbb{I}^s = \{0, 1\}$ and payoffs $u^s(0,0) = 1$, $u^s(0,1) = 1.5$, $u^s(1,0) = 1.1$, and $u^s(1,1) = 1.2$ for $s \in \{a, b\}$. These payoffs satisfy Common Ranking but not Supermodularity. With matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ on the unit simplex ($\lambda^a \geq 0$, $\lambda^b \geq 0$, and $\lambda^a + \lambda^b \leq 1$), market utilities $\bar{U}^s(0) = 0.33$ and $\bar{U}^s(1) = 0.25$, and appropriate population distributions ($F^s(0) = 0.124$ and $c \approx 0.293$), we find a competitive search equilibrium with pooling. Specifically, a single market attracts both types on both sides, with $G^s(0) = F^s(0)$, $\lambda^s \approx 0.320$, and $\phi^s \approx 0.406$ for $s \in \{a, b\}$.

Second, the lemma establishes that only the constraint of excluding lower types can bind in equilibrium:

Corollary 2 *Assume Common Ranking and Supermodularity. In any competitive search equilibrium, take an active market $\tau = (\phi^s, G^s)_{s=a,b} \in T$ and the type i in the support of G^s. Then for all $i' > i$, $\bar{U}^s(i') > U^s(i', \tau, \Lambda^s(\tau))$.*

This slack constraint for excluding higher types proves especially valuable for characterizing equilibria with positive assortative matching, as it allows us to solve for market utilities recursively starting from the lowest type.

4.4 Monotonicity of Market Utility

To further characterize equilibrium, we impose monotonicity in agents' willingness-to-pay for matching. This additional structure ensures that market utility varies systematically with agent types, simplifying the analysis of incentive constraints.

Assumption 3 (Increasing WTP) $\forall\, s = a, b,\ i > i' \in \mathbb{I}^s,\ j \in \mathbb{I}^{-s},\ u^s(i, j) \geq u^s(i', j).$

Assumption 4 (Decreasing WTP) $\forall\, s = a, b,\ i > i' \in \mathbb{I}^s,\ j \in \mathbb{I}^{-s},\ u^s(i, j) \leq u^s(i', j).$

We refer to these assumptions collectively as monotone willingness to pay.

Under either assumption, market utilities inherit the monotonicity of the underlying preferences:

Lemma 4 *Assume Common Ranking. In any competitive search equilibrium:*

1. *If willingness-to-pay is increasing, then $\bar{U}^s(i)$ is weakly increasing in i.*
2. *If willingness-to-pay is decreasing, then $\bar{U}^s(i)$ is weakly decreasing in i.*

The proof is again in Appendix A. The proof essentially works by eliminating the fees from problem (7) or, in the Supermodular case, from problem (8):

$$\max_{(\lambda^a, \lambda^b) \in \mathbb{\Lambda}, (k^a, k^b)} m(\lambda^a, \lambda^b) \left(u^a(k^a, k^b) + u^b(k^b, k^a) - \frac{\bar{U}^a(k^a)}{\lambda^a} - \frac{\bar{U}^b(k^b)}{\lambda^b} \right) \tag{9}$$
$$\text{s.t. } \bar{U}^s(i) - \bar{U}^s(k^s) \geq \lambda^s (u^s(i, k^{-s}) - u^s(k^s, k^{-s}) \ \forall i < k^s, i \in \mathbb{I}^s, s \in \{a, b\},$$
$$u^a(k^a, k^b) + u^b(k^b, k^a) - \frac{\bar{U}^a(k^a)}{\lambda^a} - \frac{\bar{U}^b(k^b)}{\lambda^b} \geq 0.$$

The remaining incentive constraint then determines monotonicity of market utility. This monotonicity result provides the foundation for our subsequent analysis of equilibrium matching patterns, where we turn to continuous types.

5 Continuous Type Distribution

The previous sections established fundamental properties of equilibrium in our two-sided matching market with private information: equilibria must be separating under supermodularity, market utilities are continuous and monotonic, and the platform's problem can be reformulated as a constrained optimization. We now leverage these insights to provide a detailed characterization of equilibrium when agent types are continuously distributed. This continuous framework allows us to employ differential methods, yielding sharp predictions about equilibrium structure and matching patterns. We start by developing a first-order approach that replaces global incentive constraints with local ones, showing this simplification is valid under our maintained assumptions. We then exploit the resulting differential structure to characterize positively assortative matching (PAM) equilibria, with particularly clean results emerging in the symmetric case. Finally, we establish sufficient conditions for the existence of PAM equilibria in a limiting case without search frictions.

5.1 First-order Approach

Having established that all equilibrium markets must be separating under Common Ranking and Supermodularity (Lemma 3), we now develop tools for characterizing these markets when types are continuously distributed. Throughout this section, we focus on the case where the support of the type distribution is the unit interval, $\mathbb{I}^s = [0, 1]$ for $s \in \{a, b\}$.

Our first step is to establish sufficient regularity of market utilities to justify a local approach to incentive constraints:

Lemma 5 *Assume Common Ranking, Supermodularity, and Monotone Willingness-to-Pay. If $\mathbb{I}^s = [0, 1]$ for $s = a, b$, then in any competitive search equilibrium, market utilities $\bar{U}^s(i)$ are differentiable almost everywhere. For any active market $\tau \in T$ that attracts types (k^a, k^b) with U^s differentiable at k^s,*

$$\bar{U}^{a\prime}(k^a) = \Lambda^a(\tau)u_1^a(k^a, k^b) \quad \text{and} \quad \bar{U}^{b\prime}(k^b) = \Lambda^b(\tau)u_1^b(k^b, k^a),$$

where u_1^s denotes the partial derivative of u^s with respect to its first argument.

The proof is in Appendix B.

Given this lemma, for any market attracting types k^a and k^b with U^s differentiable at

k^s, we can replace the global incentive constraints with local ones in problem (9).

$$\hat{V}(k^a, k^b) = \max_{(\lambda^a,\lambda^b)\in\mathbb{\Lambda}} m(\lambda^a, \lambda^b) \sum_{s=a,b} \left(u^s(k^s, k^{-s}) - \frac{\bar{U}^s(k^s)}{\lambda^s} \right) \tag{10}$$
$$\text{s.t. } \bar{U}^{s\prime}(k^s) = \lambda^s u_1^s(k^s, k^{-s}) \text{ for } s = a, b$$
$$\sum_{s=a,b} \left(u^s(k^s, k^{-s}) - \frac{\bar{U}^s(k^s)}{\lambda^s} \right) \geq 0.$$

If the constraint set is empty, $\hat{V}(k^a, k^b)$ is undefined. We stress that $\hat{V}(k^a, k^b)$ represents the gross profits a platform earns from posting a terms-of-trade that matches k^a to k^b, assuming that only the local incentive constraint is binding. Even if (k^a, k^b) yields a high value of $\hat{V}(k^a, k^b)$, creating this market may not be feasible because other incentive constraints may bind.

We can use the constraints in problem (10) to eliminate for λ^s from the problem, expressing the platform's maximized payoff directly:

$$\hat{V}(k^a, k^b) = m\left(\frac{\bar{U}^{a\prime}(k^a)}{u_1^a(k^a, k^b)}, \frac{\bar{U}^{b\prime}(k^b)}{u_1^b(k^b, k^a)} \right) \sum_{s=a,b} \left(u^s(k^s, k^{-s}) - \frac{\bar{U}^s(k^s) u_1^s(k^s, k^{-s})}{\bar{U}^{s\prime}(k^s)} \right). \tag{11}$$

Again, we require that

$$\left(\frac{\bar{U}^{a\prime}(k^a)}{u_1^a(k^a, k^b)}, \frac{\bar{U}^{b\prime}(k^b)}{u_1^b(k^b, k^a)} \right) \in \mathbb{\Lambda}$$

and

$$\sum_{s=a,b} \left(u^s(k^s, k^{-s}) - \frac{\bar{U}^s(k^s) u_1^s(k^s, k^{-s})}{\bar{U}^{s\prime}(k^s)} \right) \geq 0;$$

if not $\hat{V}(k^a, k^b)$ is undefined.

Equation (11) captures how local incentive constraints shape the platform's profits. The matching rates are determined by the elasticity of agents' market utility $\bar{U}^s$ relative to their payoff u^s, while the total surplus must cover both the direct utility from matching and an information rent term reflecting agents' local incentive constraints. This expression reveals important features of the equilibrium construction. When $k^a, k^b > 0$ and market utilities $\bar{U}^s(i)$ are known, $\hat{V}(k^a, k^b)$ is completely determined by incentive constraints. There is no optimization problem left to solve. The local incentive constraints pin down the matching rates, and these in turn determine the platform's revenue through the matching probability m. Thus, finding active markets reduces to identifying pairs (k^a, k^b) that yield the highest value of $\hat{V}(k^a, k^b)$, subject to market clearing.

However, some challenges remain. First, we need to determine the market utilities $\bar{U}^s(i)$

themselves. Second, markets that attract the lowest types ($k^a = 0$ or $k^b = 0$) require special treatment because the local incentive constraints don't bind for these types. These markets must solve the unconstrained platform problem, which then provides boundary conditions for the market utilities of higher types. Similarly, we can only compute $\hat{V}(k^a, k^b)$ at points where the U^s are differentiable, which we have so far established is true almost everywhere. Lastly, we need to verify that the sum of fees is non-negative at all active terms-of-trade. This requirement is not relevant when $c > 0$ since all platforms must make strictly positive profits to break even. When $c = 0$, this requirement becomes relevant, which we discuss wherever it is necessary.

This structure suggests a recursive approach to finding equilibria: solve the unconstrained problem for the lowest types, use this to initiate the market utilities, and then ensure higher-type markets satisfy both the local incentive constraints and market clearing. We pursue this approach in detail when we study positively assortative matching in the remainder of this section.

5.2 Necessary Conditions for PAM under Symmetry

In this section, we study a particular class of equilibria that combine positive assortative matching with symmetry between sides of the market. This focus serves two purposes. First, such equilibria are particularly tractable as the analysis reduces to a one-dimensional problem. Second, many real-world matching markets exhibit substantial symmetry between sides (for example, dating markets or peer-to-peer trading platforms), making this a natural case to understand thoroughly.

We impose symmetry through four key assumptions:

- Equal populations on each side: $I^a = I^b = I$;
- Identical type distributions: $F^a = F^b = F$;
- Symmetric match utilities: $u^a(i, j) = u^b(j, i) = u(i, j)$ for all i, j, satisfying Common Ranking, Supermodularity, and Monotone WTP;
- Symmetric matching technology: $m(\lambda^a, \lambda^b) = m(\lambda^b, \lambda^a)$.

We then look for a symmetric competitive search equilibrium with positively assortative matching, meaning it satisfies three properties:

- Side a type i agents match with side b type i agents for all i;
- Same-type agents on different sides have the same market utility: $\bar{U}^a(i) = \bar{U}^b(i) = \bar{U}(i)$;

- Markets for same-type agents have symmetric terms: In the active market τ that attracts type i agents on both sides, we have $\lambda^a(\tau) = \lambda^b(\tau) = \ell(i)$ and fees $\phi^a(\tau) = \phi^b(\tau) = \Phi(i)$.

Such equilibria, if they exist, exhibit PAM. Moreover, symmetry reduces our search for equilibrium to a one-dimensional problem: once we determine the market utility $\bar{U}(i)$ for each type i, the matching rates and fees for both sides of each market follow directly. Additionally, rather than working with the two dimensional feasible set of matching probabilities $\mathbb{\Lambda}$, we can work with the one-dimensional set $[0, \bar{\lambda}]$, where $(\bar{\lambda}, \bar{\lambda}) \in \mathbb{\Lambda}$ and satisfies $m(\bar{\lambda}, \bar{\lambda}) = 0$. Thus $\bar{\lambda}$ is the highest matching probability that can be obtained by both sides of the market under symmetry.

For any market attracting type i agents from both sides, define:

$$h(\lambda, i) \equiv \lambda \left(u(i,i) - \frac{c}{2m(\lambda, \lambda)} \right). \tag{12}$$

This captures the per-person utility in a symmetric market where i matches with i with probability $\lambda \in [0, \bar{\lambda}]$: $u(i,i)$ is the utility from matching with the same type, and $\frac{c}{2m(\lambda,\lambda)}$ is the fee that each party must pay to cover the platform's advertising cost c.[2]

Under full information (ignoring incentive constraints), the equilibrium matching probability for type i agents would maximize $h(\lambda, i)$ subject to the nonnegativity constraint on fees, $u(i,i) - \frac{c}{2m(\lambda,\lambda)} > 0$. Let $\ell^*(i)$ denote this maximizer. For expositional convenience, we assume in this section that $\ell^*(i) > 0$ for all i, or equivalently $2u(i,i)m(0,0) > c$; otherwise some types would not participate in the market both with private and full information. Because only downward incentive constraints can bind (from Lemma 3 and Corollary 2), the lowest types ($i = 0$) must achieve this unconstrained optimum: $\ell(0) = \ell^*(0)$.

For higher types, however, local incentive constraints bind. In any active market, agents must receive utility $\bar{U}(i) = h(\ell(i), i)$. Differentiating this equation using the definition of h in equation (12) gives

$$\bar{U}'(i) = h_1(\ell(i), i)\ell'(i) + \ell(i)\big(u_1(i,i) + u_2(i,i)\big),$$

where h_1 denotes the partial derivative of h with respect to its first argument. Moreover, from Lemma 5, we know that the local incentive constraint $\bar{U}'(i) = \ell(i)u_1(i,i)$ holds almost

[2]For $c > 0$, h is always defined for $\lambda \in [0, \bar{\lambda})$ but converges to $-\infty$ as $\lambda \to \bar{\lambda}$. When $c = 0$, $h(\lambda, i) = \lambda u(i,i)$, so h is defined on the closed interval $[0, \bar{\lambda}]$.

everywhere. Plugging into the previous equation, we get that

$$h_1(\ell(i), i)\frac{\ell'(i)}{\ell(i)} = -u_2(i, i) \tag{13}$$

almost everywhere.

This ordinary differential equation (13), combined with the initial condition $\ell(0) = \ell^*(0)$, characterizes how matching probabilities must vary with types in any symmetric PAM equilibrium. The structure of the equation reveals how incentive constraints distort matching probabilities relative to the full information benchmark $\ell^*(i)$. The left-hand side captures the rate of change in matching probabilities, scaled by h_1 which measures the marginal value of increasing the meeting probability. The right-hand side involves $u_2(i, i)$, which captures how an agent's utility changes when their partner's type increases.

This discussion glosses over four technical issues. First, at $i = 0$, the optimization problem is unconstrained, so the first order condition $h_1(\ell(0), 0) = 0$ holds. This in turn means that we cannot directly apply the ordinary differential equation (13) at $i = 0$. Instead, we look for a solution to the differential equation with the property that $\lim_{i\to 0} \ell(i) = \ell^*(0)$.

Second, Lemma 4 tells us that market value is monotone and so almost everywhere differentiable. This means that the differential equation (13) only holds almost everywhere, leaving open the possibility of discontinuities in ℓ function. The proof of Proposition 1 establishes that ℓ must in fact be continuous.

Third, there are two such solutions, one with ℓ increasing and one with ℓ decreasing. We show in the proof of Proposition 1 that the first solution is valid with Increasing WTP and the second with Decreasing WTP.

Lastly, is it possible that the solution to the differential equation (13), combined with the boundary conditions, give us matching probabilities and equilibrium utilities that are non-positive? We show this can never be the case in the proof of Proposition 1. These technical considerations, while subtle, are crucial for a complete understanding of the equilibrium structure. They highlight how the matching technology's properties interact with agents' preferences to determine the appropriate solution concept.

The following Proposition combines these insights:

Proposition 1 *Assume Common Ranking, Supermodularity, and Monotone WTP. Also assume $2u(i, i)m(0, 0) > c$ and $\frac{u_2(i,i)}{u(i,i)}$ is bounded above, for all i. If there exists a symmetric competitive search equilibrium with PAM, then:*

1. *The matching probabilities $\ell(i)$ solve ordinary differential equation (13) with limiting condition $\ell(0) = \ell^*(0)$;*

2. *Market utilities satisfy* $\bar{U}(i) = h(\ell(i), i)$*;*

3. *Fees satisfy* $\Phi(i) = u(i,i) - \bar{U}(i)/\ell(i)$*;*

4. *These solutions satisfy all global incentive constraints.*

Moreover, with

- *Increasing WTP,* $\bar{U}(i)$*,* $\ell(i)$*, and* $\Phi(i)$ *are increasing in* i *and* $\ell(i) \in [\ell^*(i), \bar{\lambda}]$ *for all* i*;*
- *Decreasing WTP,* $\bar{U}(i)$*,* $\ell(i)$*, and* $\Phi(i)$ *are decreasing in* i *and* $\ell(i) \in (0, \ell^*(i)]$ *for all* i*.*

The proof is in Appendix B.

Proposition 1 has several notable implications. First, the differential equation characterization of equilibrium has remarkable simplicity and power. While we only imposed local incentive constraints, the resulting allocation automatically satisfies global incentive constraints. This is far from obvious: in many mechanism design problems, local incentive constraints are not sufficient for global incentive compatibility.

Second, the direction of Monotone Willingness-to-Pay completely determines the structure of equilibrium utilities, contact rates, and fees. With Increasing WTP, higher types receive higher utilities and face higher contact rates and fees. With Decreasing WTP, the pattern reverses. This stands in contrast to the case with observable types (see Section 7.2), where the direction of these patterns depends on whether the equilibrium payoff $u(i,i)$ is increasing or decreasing, not on the willingness-to-pay.

Third, private information systematically distorts allocations relative to the observable types benchmark. With Increasing WTP, equilibrium contact rates exceed the full-information optimum $\ell^*(i)$. The free entry condition then implies that fees must also be higher to cover platform costs. With Decreasing WTP, these distortions reverse: both contact rates and fees fall below their full-information levels. These distortions reflect platforms' optimal response to incentive constraints: they adjust contact rates and fees to make mimicking less attractive to lower types.

We close this section by noting that, while Proposition 1 provides a complete characterization of symmetric PAM equilibria when they exist, it does not guarantee existence. Once we find candidate equilibrium utilities $\bar{U}(i)$ by solving the ODE, we know all incentive constraints will be satisfied, both local and global. However, we still must verify that no platform would prefer to create a market matching different types. Since we know the market utility in this candidate equilibrium, this amounts to checking the value of objects like $\hat{V}(k^a, k^b)$ in equation (11), which is straightforward but must be done numerically. The next subsection tackles this verification problem directly, providing sufficient conditions for the existence of PAM equilibria.

5.3 Sufficient Conditions for PAM under Symmetry

We consider a limit without search frictions. More specifically, we consider the special case where $c = 0$. We study two cases, first with an increasing WTP and then with a decreasing WTP. The characterization in the two cases is notably different, an issue we return to at the end of this section.

Assumption 5

(a) $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}\right) \notin \mathbb{A}$ *for* $i \neq j$,

(b) $u_1(i,0) = 0$, *for* $i \in \mathbb{I}$.

Proposition 2 *Assume Common Ranking, Supermodularity, and increasing WTP. Also assume* $c = 0$ *and assumption 5, there is a PAM competitive search equilibrium. In the market attracting type* i *agents, the matching probability is* $\ell(i) = \bar{\lambda}$*; the fee solves* $\Phi'(i) = u_2(i,i)$ *with* $\Phi(0) = 0$*; equilibrium utility solves* $\bar{U}'(i) = \bar{\lambda}u_1(i,i)$ *with* $\bar{U}(0) = \bar{\lambda}u(0,0)$.

The proof is in Appendix B.

Assumption 6

(a) $u(i,j) + u(j,i) \leq u(i,i)\frac{u_1(i,j)}{u_1(i,i)} + u(j,i)\frac{u_1(j,i)}{u_1(j,j)}$

(b) $\left(\bar{\lambda}\frac{u(0,0)u_1(j,j)}{u(j,j)u_1(j,0)}\exp\left(\int_0^j \frac{u_1(k,k)}{u(k,k)}dk\right), \frac{u(0,j)+u(j,0)-u(j,j)\frac{u_1(j,0)}{u_1(j,j)}}{\bar{\lambda}u(0,0)}\right) \notin \mathbb{A}$.

Proposition 3 *Assume Common Ranking, Supermodularity, and Decreasing WTP. Also assume* $c = 0$ *and assumption 6, there is a PAM competitive search equilibrium. In the market attracting type* i *agents, the matching probability solves* $\ell'(i) = -\frac{u_2(i,i)}{u(i,i)}$, *with* $\ell(0) = \bar{\lambda}$ *and the fee is* $\Phi(i) = 0$, $\forall i$*; equilibrium utility solves* $\frac{\bar{U}'(i)}{\bar{U}(i)} = \frac{u_1(i,i)}{u(i,i)}$, *with* $\bar{U}(0) = \bar{\lambda}u(0,0)$.

The proof is in Appendix B.

Again, we illustrate this proposition with a simple example. Let $u(i,j)$ satisfy

$$u(i,j) = 1 - \kappa\Phi(i)(1 - \phi(j))$$

for $0 \leq \Phi(i) \leq 1$ and $0 < \kappa \leq 1$. We think of this as a model of disease transmission. i is the probabliity that an individual is healthy. If he interacts with anyone, he gets a utility benefit of 1, but if he interacts with a sick person and was previously healthy, he gets the disease, costing utility κ. Thus the cost of interacting with someone who is healthy with probability j is $\kappa i(1-j)$. Again, this function satisfies Common Ranking, Supermodularity, and Decreasing WTP, as well as conditions (1) in assumption (6). It satisfies the condition (2) in assumption (6) if $\bar{\lambda} = 1$. The proposition then establishes that there is a PAM competitive search equilibrium with this payoff function and characterizes it completely.

There are two ways markets can induce agents to self-select, through fees and contact rates. We find that with Increasing WTP, fees are critical in the limit without platforms. All active markets have the same contact rate, but the more desirable agents, who have a higher willingness-to-pay for matches, exclude lower types by paying higher fees. Contact rates are not useful for excluding lower types because higher types value meetings more than lower types do.

In contrast, with a Decreasing WTP, contact rates play the critical role. Now fees are zero in all markets, but the more desirable agents, who have a lower willingness-to-pay for matches, exclude lower types by suppressing their contact rate. Now fees are not useful for excluding lower types because higher types are unwilling to pay high fees for matching.

6 Other Equilibrium Patterns

6.1 PAM Equilibrium without Symmetry

In this section, we discuss an algorithm to compute a PAM competitive search equilibrium with asymmetry. For exposition simplicity, we focus on cases where all types participate: the condition (2) in the definition of a competitive search equilibrium holds with equality for types. We discuss the other cases briefly at the end of this section. Our discussion so far established that all equilibrium terms-of-trade must be separating. With PAM CSE and continuous type distribution, there is only one terms-of-trade that attracts a-side type-i agent. In this section, we discuss an algorithm to compute a competitive search equilibrium with PAM. For exposition simplicity, we focus on cases where all types participate: the condition (2) in the definition of a competitive search equilibrium holds with equality for types. We discuss the other cases briefly at the end of this section.

Our characterization follows two steps. First, in a PAM equilibrium, for the type 0's market to clear, $\sigma(0) = 0$. That is, some platforms post terms-of-trade that attracts the lowest types on both sides. From Lemma 3, this terms-of-trade is undistortted. Lemma 6 summarizes the details of this terms-of-trade. Secondly, we solve other equilibrium terms-of-trade as solution of a system of odes, detailed in Lemma 8.

Lemma 6 *Assume Common Ranking and Supermodularity. In a PAM CSE, $\sigma(0) = 0$ and $(\lambda^s(0))_{s=a,b}, \bar{U}^b(0)$ solve the following unconstrained problem, for a fixed $\bar{U}^a(0) = \upsilon$:*

$$c = \max_{\lambda^a, \lambda^b} m(\lambda^a, \lambda^b) \left(u^a(0,0) + u^b(0,0) - \frac{\upsilon}{\lambda^a} - \frac{\bar{U}^b(0)}{\lambda^b} \right), \tag{14}$$

with the solution denoted as $(\lambda_0^{s*}(\upsilon))_{s=a,b}$ and $\bar{U}_0^{b*}(\upsilon)$.

Lemma 7 *Assume common ranking, supermodularity, and monotone willingness-to-pay. In any PAM competitive search equilibrium, $\lambda^s(i)$ is continuous.*

Proof. In Appendix C.1. ∎

We characterize the PAM equilibrium using a system of odes in (λ^a, λ^b), summarized in Lemma 8.

Lemma 8 *Assume common ranking, supermodularity and continuous type distribution. A PAM CSE is characterized by $(\lambda^s, \bar{U}^s)_{s=a,b}$, σ that solve*

$$\nu^a(i) \frac{\lambda^{a\prime}(i)}{\lambda^a(i)} = -\iota(i) - u_2^b(\sigma(i), i), \qquad \nu^b(i) \frac{\lambda^{b\prime}(i)}{\lambda^b(i)} = \iota(i) - \sigma'(i) u_2^a(i, \sigma(i)), \tag{15}$$

$$\bar{U}^{a\prime}(i) = \lambda^a(i) u_1^a(i, \sigma(i)), \qquad \bar{U}^{b\prime}(\sigma(i)) = \lambda^b(i) u_1^b(\sigma(i), i), \tag{16}$$

$$\sigma'(i) = \frac{\lambda^a(i)}{\lambda^b(i)} \frac{I^a \bar{g}^a(i)}{I^b \bar{g}^b(\sigma(i))}, \tag{17}$$

with the boundary conditions

$$\sigma(0) = 0,\ \sigma(1) = 1,\ \lambda^s(0) = \lambda_0^{s*}(\bar{U}^a(0)), \bar{U}^b(0) = \bar{U}_0^{b*}(\bar{U}^a(0)),$$

where

$$\nu^a(i) = \left(\frac{\epsilon_1(\lambda^a(i), \lambda^b(i))}{m(\lambda^a(i), \lambda^b(i))} c + \frac{\bar{U}^a(i)}{\lambda^a(i)} \right),\ \nu^b(i) = \left(\frac{\epsilon_2(\lambda^a(i), \lambda^b(i))}{m(\lambda^a(i), \lambda^b(i))} c + \frac{\bar{U}^b(\sigma(i))}{\lambda^b(i)} \right), \tag{18}$$

$$\iota(i) = \sigma'(i) \nu^a(i) \frac{u_{1,2}^a(i, \sigma(i))}{u_1^a(i, \sigma(i))} - \nu^b(i) \frac{u_{1,2}^b(\sigma(i), i)}{u_1^b(\sigma(i), i)},$$

$$\epsilon_1(\lambda^a, \lambda^b) = \frac{\lambda^a m_1(\lambda^a, \lambda^b)}{m(\lambda^a, \lambda^b)},\ \ \epsilon_2(\lambda^a, \lambda^b) = \frac{\lambda^b m_2(\lambda^a, \lambda^b)}{m(\lambda^a, \lambda^b)}.$$

Proof. In Appendix C.2. ∎

Discussion. It is possible that there is no solution to the differential equation system from Lemma 8 where $\bar{U}^s(i) \geq 0$ for all $i \in \mathbb{I}^s$. In this case, the PAM equilibrium involves some types not participating. Under increasing willingness-to-pay, there are three possible non-participating scenarios. First, it could be that the side-a low types are not participating: there is a threshold type $\underline{i}^a$ such that $\bar{U}^a(i) = 0$ for all $i \leq \underline{i}^a$. In this case, the assignment function starts from $\underline{i}^a$. The second case invovles the same pattern, on side b. Lastly, it is possible that the lowest types on both sides do not participate. This happens when non of the terms-of-trade that attract side-s type-0 delivers.

6.2 NAM Equilibrium

In this section, we discuss an algorithm to compute a NAM competitive search equilibrium. First, we focus on characterization of the $(0,1)$ market. In this market, the incentive constraint on side-a is irrelevant. Given the equilibrium utility $\bar{U}^b(1)$ and λ^b, the matching proability for side-a type-0 and her equilibrium utility are solutions to:

$$c = \max_{\lambda^a} m\left(\lambda^a, \lambda^b\right)\left(u^a(0,1) + u^b(1,0) - \frac{\bar{U}^a(0)}{\lambda^a} - \frac{\bar{U}^b(1)}{\lambda^b}\right). \tag{19}$$

Lemma 9 *Assume common ranking, supermodularity and continuous type distribution. A NAM CSE is characterized by $(\lambda^s, \bar{U}^s)_{s=a,b}$, σ that solve*

$$\nu^a(i)\frac{\lambda^{a'}(i)}{\lambda^a(i)} = -\iota(i) - u_2^b(\sigma(i), i), \qquad \nu^b(i)\frac{\lambda^{b'}(i)}{\lambda^b(i)} = \iota(i) - \sigma'(i)u_2^a(i,\sigma(i)), \tag{20}$$

$$\bar{U}^{a'}(i) = \lambda^a(i)u_1^a(i,\sigma(i)), \qquad \bar{U}^{b'}(\sigma(i)) = \lambda^b(i)u_1^b(\sigma(i), i), \tag{21}$$

$$\sigma'(i) = -\frac{\lambda^a(i)}{\lambda^b(\sigma(i))}\frac{I^a\bar{g}^a(i)}{I^b\bar{g}^b(\sigma(i))}, \tag{22}$$

with the boundary conditions

$$\sigma(0) = 1,\ \sigma(1) = 0,\ (\lambda^s(0), \bar{U}^s(0)) \text{ solve problem 19 given } (\lambda^{-s}(1), \bar{U}^{-s}(1)).$$

where

$$\nu^a(i) = \left(\frac{\epsilon_1(\lambda^a(i),\lambda^b(i))}{m(\lambda^a(i),\lambda^b(i))}c + \frac{\bar{U}^a(i)}{\lambda^a(i)}\right),\ \nu^b(i) = \left(\frac{\epsilon_2(\lambda^a(i),\lambda^b(i))}{m(\lambda^a(i),\lambda^b(i))}c + \frac{\bar{U}^b(\sigma(i))}{\lambda^b(i)}\right), \tag{23}$$

$$\iota(i) = \sigma'(i)\nu^a(i)\frac{u_{1,2}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))} - \nu^b(i)\frac{u_{1,2}^b(\sigma(i),i)}{u_1^b(\sigma(i),i)},$$

$$\epsilon_1(\lambda^a,\lambda^b) = \frac{\lambda^a m_1(\lambda^a,\lambda^b)}{m(\lambda^a,\lambda^b)},\ \ \epsilon_2(\lambda^a,\lambda^b) = \frac{\lambda^b m_2(\lambda^a,\lambda^b)}{m(\lambda^a,\lambda^b)}.$$

7 Discussion

7.1 Sorting with Conventional Matching Function

The key novelty of the environment in this paper is the assumption of a matching function that with two sides of a match which can have different matching probabilities. This assumption allows for the technological possibility of negative sorting. Before proceeding to the characterization of equilibrium, we consider an alternative "random" matching technology that involves only agents and platform advertising efforts. Platforms intermediate matches between two agents. Instead of the matching function in section 2.2, suppose the number of matches in the market is $\tilde{M}(n^a, \alpha)$, where n^a is the measure of agents, and α is the effort by platforms. When a match happens, it is a match between two randomly selected agents on the platform, so if there are a shares ω_i of type i agents in a particular market, the share of (i, j) matches is $\omega_i \omega_j$. This matching technology is a direct extension of the standard bilateral matching technology into our environment.

The core result of this section is the impossibility of negative sorting with this random matching technology. To show this, we define a set of active terms-of-trade as T, and the effort by platforms as α, where $\alpha(\tau)$ is the aggregate effort across all platforms in $\tau \in T$. We define the covariance of matching types as:

$$\mathbf{COV}(T, \alpha) = \frac{\sum_{\tau \in T} \tilde{M}(N^a(\tau), \alpha(\tau)) \sum_{i=1}^{I} \sum_{j=1}^{J} \omega_i \omega_j \left(i - \mathbf{E}(i)\right) \left(j - \mathbf{E}(j)\right)}{\sum_{\tau \in T} \tilde{M}(N^a(\tau), \alpha(\tau))}, \tag{24}$$

where

$$\mathbf{E}(i) = \frac{\sum_{\tau \in T} \tilde{M}(N^a(\tau), \alpha(\tau)) \sum_{i=1}^{I} \omega_i i}{\sum_{\tau \in T} \tilde{M}(N^a(\tau), \alpha(\tau))} \quad \text{and} \quad \mathbf{E}(j) = \frac{\sum_{\tau \in T} \tilde{M}(N^a(\tau), \alpha(\tau)) \sum_{i=1}^{I} \omega_i j}{\sum_{\tau \in T} \tilde{M}(N^a(\tau), \alpha(\tau))}.$$

The covariance between agents' types and their matching partners' types must be non-negative with random matching technology, as stated in the following lemma:

Lemma 10 *Under a random matching technology,* $\mathbf{COV}(T, \alpha) \geq 0$.

Proof. Due to random matching within markets, $\mathbf{E}(i) = \mathbf{E}(j)$. Thus we can write the covariance as

$$\mathbf{COV}(T, \alpha) = \frac{\sum_{\tau \in T} \tilde{M}(N^a(\tau), \alpha(\tau)) \left[\sum_{i=1}^{I} \omega_i \left(i - \mathbf{E}(i)\right)\right]^2}{\sum_{\tau \in T} \tilde{M}(N^a(\tau), \alpha(\tau))} \geq 0.$$

■

Lemma 10 highlights a physical restriction on sorting pattern due to random matching within markets. It is impossible to have negative assortative matching because for each match made between a high-type agent and a low-type agent, there must be another match between each types within themselves. The core assumption that leads to this result is that the contact rates for agents within each market equal. To allow for the possibility of negative assortative matching, we introduce our matching technology with two sides and a roll for platforms.

7.2 Sorting with Observable Types

First consider the competitive search equilibrium with observable types. This environment is similar to Shi (2001) and Eeckhout and Kircher (2010). The recast of their result in our definition will be a benchmark for our characterization of competitive search equilibrium with private information. In the observable-type model, the platforms can directly control the type distributions by charging type-specific fees. We omit the details for the definition of equilibrium with observable types in Appendix C.3.

Proposition 4 *Assume* $M\left(n^a, n^b\right) = \left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-\frac{1}{\gamma}}$ *and* $f_{i,j} \neq f_{i,j'}$ *for* $j \neq j'$. *There exists a unique PAM competitive search equilibrium with observable types if* $f_{i,j}^{\frac{\gamma}{1+\gamma}}$ *is supermodular.*

Proof. In Appendix C.4. ■

8 Concluding Remarks

How do individuals sort in the presence of private information? We use a competitive search model to answer that question. We prove that under Common Ranking and Supermodularity, each side of every active market only attracts one type of agent. We then characterize positively assortative matching under those two assumptions and Monotonic Willingness-to-Pay. We prove that with Increasing WTP, higher types have higher utility, match at a faster rate, and pay higher fees. With Decreasing WTP, higher types have lower utility, match at a lower rate, and pay lower fees. Finally, we provide sufficient conditions for positively assortative matching when the role of platforms vanishes. With Increasing WTP, only fees differ across active markets, and the matching rate is driven to 1. With Decreasing WTP, only matching rates differ across markets, and fees are driven to zero.

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Appendix

A Proofs for Section 4

Proof of Lemma 1. Suppose not, so $\bar{U}^s$ is discontinuous at some point $i^* \in [0,1]$. Then there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exist a point i with $|i - i^*| < \delta$ and $|\bar{U}^s(i) - \bar{U}^s(i^*)| > \epsilon$.

First assume it is possible to find such a point with $\bar{U}^s(i^*) - \bar{U}^s(i) > \epsilon$. By the third condition in the definition of competitive search equilibrium, type i^* must obtain utility $\bar{U}^s(i^*) = \Lambda^s(\tau)(\int u^s(i^*, j)dG^{-s}(j) - \phi^s)$ in some market with terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b} \in T$.

Continuity of u^s in its first argument implies we can choose δ sufficiently small such that $\int u^s(i^*, j)dG^{-s}(j) - \int u^s(i, j)dG^{-s}(j) < \epsilon$. Since $0 \le \Lambda^s(\tau) \le 1$, this means

$$\bar{U}^s(i) < \bar{U}^s(i^*) - \epsilon = \Lambda^s(\tau)\left(\int u^s(i^*, j)dG^{-s}(j) - \phi^s\right) - \epsilon < \Lambda^s(\tau)\left(\int u^s(i, j)dG^{-s}(j) - \phi^s\right).$$

The first inequality comes from the assumed discontinuity in $\bar{U}^s$. The equation is the indifference condition of i^*. The second inequality uses the continuity of u^s. But this implies that the terms-of-trade τ do not satisfy condition 1(a) in the definition of partial equilibrium for the type i agent, a contradiction.

If instead we have $\bar{U}^s(i) - \bar{U}^s(i^*) > \epsilon$, we reverse the role of i and i^* in the proof, but the argument is otherwise unchanged. ■

Proof of Lemma 2. First, we verify that the construction yields a partial equilibrium. We check each condition of Definition 1:

1. (Optimal Search) Take any $\tau \in T^p$. By construction:
   (a) $\bar{U}^s(i) \ge U^s(i, \tau, \Lambda^s(\tau))$ for all $i \in \mathbb{I}^s$, since this is exactly the first constraints of Problem (7);
   (b) $\int_{\mathbb{I}^s} \bar{U}^s(i)dG^s(i) = \int_{\mathbb{I}^s} U^s(i, \tau, \Lambda^s(\tau))dG^s(i)$, from the second constraints.
2. (Impossible Terms-of-Trade) For any $\tau \notin T^p$, by construction there exist no $(\lambda^s)_{s=a,b} \in \mathbb{\Lambda}$ satisfying the constraints of Problem (7). This directly implies there are no $(\lambda^s)_{s=a,b}$ satisfying conditions 2(a) and 2(b) of Definition 1
3. (Profit Maximization) Take any $\tau \in T^p$ with $\tau \notin T$. By construction, $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ does not solve Problem (7). Therefore, there exists some $(\hat{\lambda}^s, \hat{\phi}^s, \hat{G}^s)_{s=a,b}$ satisfying the

constraints with strictly higher objective value. Let $\hat{\tau} = (\hat{\phi}^s, \hat{G}^s)_{s=a,b}$, so $\Lambda^s(\hat{\tau}) = \hat{\lambda}^s$. Then $\hat{\tau} \in T^p$ and $V(\hat{\tau}, \Lambda^a(\hat{\tau}), \Lambda^b(\hat{\tau})) > V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$

For the converse, take any partial equilibrium $\{T^p, T, (\Lambda^s, \bar{U}^s)_{s=a,b}\}$ and any $\tau = (\phi^s, G^s)_{s=a,b} \in T$. We must show that $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ solves Problem (7). From Definition 1 parts 1(a) and 1(b), this tuple satisfies the constraints of Problem (7). Suppose it is not optimal. Then there exists some $(\hat{\lambda}^s, \hat{\phi}^s, \hat{G}^s)_{s=a,b}$ satisfying the constraints with strictly higher objective value. Let $\hat{\tau} = (\hat{\phi}^s, \hat{G}^s)_{s=a,b}$. The constraints of Problem (7) imply $\hat{\tau} \in T^p$ with $\Lambda^s(\hat{\tau}) = \hat{\lambda}^s$. But then $V(\hat{\tau}, \Lambda^a(\hat{\tau}), \Lambda^b(\hat{\tau})) > V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$, so part 3 of Definition 1 implies $\tau \notin T$, a contradiction. ■

Proof of Lemma 3.

Necessity. We first prove that any CSE $\left(T^p, T, (\Lambda^s, \bar{U}^s)_{s=a,b}, \mu\right)$ must satisfy these conditions. Condition 4 is a piece of the definition of CSE. Condition 3 holds in any partial equilibrium, and a CSE is a partial equilibrium. Thus Condition 3 holds as well.

Next, take any $\tau \in T$. Let $\lambda^s = \Lambda^s(\tau)$. Lemma 2 implies that $(\lambda^s, \phi^s, G^s)_{s=a,b}$ must solve problem (7) with maximal value c (from the free entry condition in the definition of CSE). We prove the solution to problem (7) must have a degenerate distribution G^s and moreover must solve problem (8).

Suppose to the contrary that either G^a or G^b is non-degenerate. Let k_1^s be the largest s-side agent with a binding incentive constraint:

$$k_1^s = \max\left\{ i \in \mathbb{I}^s | \bar{U}^s(i) = \lambda^s \left(\int_{\mathbb{I}^{-s}} u^s(i,j) dG^{-s}(j) - \phi^s \right) \right\} \tag{25}$$

u^s is continuous in its first argument by assumption, and $\bar{U}^s$ is continuous in a CSE (Lemma 1). Then since $\mathbb{I}^s$ is compact, the type k_1^s exists. Moreover, k_1^s exceeds all points in the support of G^s, strictly so for a positive measure on at least one side given non-degeneracy.

Define new fees $\phi_1^s \equiv u^s(k_1^s, k_1^{-s}) - \bar{U}^s(k_1^s)/\lambda^s$. Common ranking implies $\phi_1^s \geq \phi^s$ with strict inequality on at least one side, proving $\phi_1^a + \phi_1^b > \phi^a + \phi^b$.

We next prove that for any $k' < k_1^s$, $\bar{U}^s(k') \geq \lambda^s(u^s(k', k_1^{-s}) - \phi^s)$:

$$\begin{aligned}
\lambda^s(u^s(k', k_1^{-s}) - \phi_1^s) &= \lambda^s \int_{\mathbb{I}^{-s}} (u^s(k', k_1^{-s}) - u^s(k', j))dG^{-s}(j) \\
&\qquad\qquad\qquad + \lambda^s \int_{\mathbb{I}^{-s}} u^s(k', j)dG^{-s}(j) - \lambda^s \phi_1^s \\
&\leq \lambda^s \int_{\mathbb{I}^{-s}} (u^s(k_1^s, k_1^{-s}) - u^s(k_1^s, j))dG^{-s}(j) \\
&\qquad\qquad\qquad + \lambda^s \int_{\mathbb{I}^{-s}} u^s(k', j)dG^{-s}(j) - \lambda^s \phi_1^s \\
&= \lambda^s \phi_1^s - \lambda^s \phi^s + \lambda^s \int_{\mathbb{I}^{-s}} u^s(k', j)dG^{-s}(j) - \lambda^s \phi_1^s \\
&\leq \bar{U}^s(k').
\end{aligned}$$

The first equality adds and subtracts $\int_{\mathbb{I}^{-s}} u^s(k', j)dG^{-s}(j)$ and reorders terms. The first inequality uses supermodularity, together with $k' < k_1^s$ and $k_1^{-s} \geq j$ for all j in the support of G^{-s}. The second equality uses the definition of ϕ_1^s as well as the binding constraint in the definition of k_1^s in equation (25). The final inequality collects terms and uses the incentive constraint in the original problem, $\bar{U}^s(k') \geq \lambda^s\big(\int_{\mathbb{I}^{-s}} u^s(k', j)dG^{-s}(j) - \phi^s\big)$.

We do not claim $\bar{U}^s(k') \geq \lambda^s(u^s(k', k_1^{-s}) - \phi^s)$ for $k' > k_1^s$. Instead, we construct monotone sequences (k_n^s, ϕ_n^s) by iteratively finding the highest type willing to pay more to match with previous highest types. Suppose for some $n \geq 2$, we have found non-decreasing sequences of types $k_1^s \leq \cdots \leq k_{n-1}^s$ as well as sequences of fees $\phi_1^s, \ldots, \phi_{n-1}^s$ with the following properties:

1. $\bar{U}^s(k_{n-1}^s) = \lambda^s\big(u^s(k_{n-1}^s, k_{n-1}^{-s}) - \phi_{n-1}^s\big)$, $s \in \{a, b\}$;

2. $\bar{U}^s(k') \geq \lambda^s\big(u^s(k', k_{n-1}^{-s}) - \phi_{n-1}^s\big)$ for all $k' < k_{n-1}^s$, $k' \in \mathbb{I}^s$, $s \in \{a, b\}$;

3. $\phi_{n-1}^a + \phi_{n-1}^b \geq \phi_1^a + \phi_1^b$.

We have done this when $n = 2$ and proceed by induction to extend this to any n.

Let

$$k_n^s \equiv \max\left(\arg\max_{k' \in \mathbb{I}^s}\left(u^s(k', k_{n-1}^s) - \frac{\bar{U}^s(k')}{\lambda^s}\right)\right). \tag{26}$$

That is, k_n^s is the largest element of the set of maximizers of $u^s(k', k_{n-1}^s) - \bar{U}^s(k')/\lambda^s$. The first two properties above imply that for all $k' < k_{n-1}^s$,

$$u^s(k', k_{n-1}^{-s}) - \frac{\bar{U}^s(k')}{\lambda^s} \leq \phi_{n-1}^s = u^s(k_{n-1}^s, k_{n-1}^{-s}) - \frac{\bar{U}^s(k_{n-1}^s)}{\lambda^s}.$$

Thus $k_n^s \geq k_{n-1}^s$. Also define

$$\phi_n^s \equiv u^s(k_n^s, k_n^{-s}) - \frac{\bar{U}^s(k_n^s)}{\lambda^s}$$

The same arguments as for $n = 1$ imply $\phi_n^s \geq \phi_{n-1}^s$ and $\bar{U}^s(k') \geq \lambda^s(u^s(k', k_n^{-s}) - \phi_n^s)$ for all $k' < k_n^s$.

The sequences (k_n^a, k_n^b) are nondecreasing on the compact set $\mathbb{I}^a \times \mathbb{I}^b$ and so converge to $(k^{a*}, k^{b*}) \in \mathbb{I}^a \times \mathbb{I}^b$. This means the ϕ^s converges as well, to (ϕ^{a*}, ϕ^{b*}), with $\phi^{a*} + \phi^{b*} \geq \phi_1^a + \phi_1^b > \phi^a + \phi^b$. Continuity of $\bar{U}^s$ and u^s implies

1. $\bar{U}^s(k^{s*}) = \lambda^s\big(u^s(k^{s*}, k^{-s*}) - \phi^{s*}\big)$, $s \in \{a, b\}$;
2. $\bar{U}^s(k') \geq \lambda^s\big(u^s(k', k^{-s*}) - \phi^{s*}\big)$ for all $k' < k^{s*}$, $k' \in \mathbb{I}^s$, $s \in \{a, b\}$;
3. $\phi^{a*} + \phi^{b*} \geq \phi_1^a + \phi_1^b > \phi^a + \phi^b$.

Finally, we claim the second point, the incentive constraint, extends to all $k' \in \mathbb{I}^s$, including $k' > k^{s*}$. To prove this, suppose to the contrary that there is a $k' > k^{s*}$ with $\bar{U}^s(k') < \lambda^s\big(u^s(k', k^{-s*}) - \phi^{s*}\big)$. Eliminate ϕ^{s*} using the first bullet point:

$$\bar{U}^s(k') < \lambda^s\big(u^s(k', k^{-s*}) - u^s(k^{s*}, k^{-s*})\big) + \bar{U}^s(k^{s*}).$$

But continuity of u^s and convergence of k_n^{-s} to k^{-s*} implies that there exists an n such that

$$\bar{U}^s(k') < \lambda^s\big(u^s(k', k_n^{-s}) - u^s(k^{s*}, k_n^{-s})\big) + \bar{U}^s(k^{s*}),$$

or equivalently

$$u^s(k', k_n^{-s}) - \frac{\bar{U}^s(k')}{\lambda^s} > u^s(k^{s*}, k_n^{-s}) - \frac{\bar{U}^s(k^{s*})}{\lambda^s},$$

From the definition of k_{n+1}^s (equation (26)), this implies $k' > k^{s*}$, which in turn contradicts monotonicity of the sequence k_n^s with convergence to k^{s*}. So we conclude that $\bar{U}^s(k') \geq \lambda^s\big(u^s(k', k^{-s*}) - \phi^{s*}\big)$ for all $k' \in \mathbb{I}^s$, $s \in \{a, b\}$.

Now let G^{s*} be the cumulative distribution function that is degenerate at k^{s*} and consider the alternative term-of-trade $(\lambda^s, \phi^{s*}, G^{s*})_{s=a,b}$. By construction this satisfies the constraints in problem (7) and it attains a higher value than the original term-of-trade because $\phi^{a*} + \phi^{b*} > \phi^a + \phi^b$ and (λ^a, λ^b) is unchanged. This proves that the original term-of-trade does not solve problem (7). That is, any solution to the problem (7) must have a degenerate type distribution on both sides of the market.

This proves the first two conditions in the statement of the Lemma, since any solution to problem (7) must have a degenerate type distribution and slack upward constraints, and thus also solve problem (8). Following the same steps, we also find that any solution to problem (8) also solves problem (7).

Sufficiency. For sufficiency, we need to prove that a tuple satisfying the four conditions constitutes both a partial equilibrium and a CSE. Let's verify each requirement:

First, for the partial equilibrium conditions from Definition 1:

1. Optimal Search and Impossible Terms-of-Trade follow directly from condition 3;

2. For Profit Maximization, we proceed in steps. Condition 1 ensures each $\tau \in T$ is separating, and condition 2 ensures it solves problem (8). Given the equivalence between problems (7) and (8) when condition 1 is satisfied, Lemma 2 allows us to map from the solution to problem (8) back to a partial equilibrium. Thus condition 2 ensures that each $\tau \in T$ satisfies the final partial equilibrium requirement, profit maximization.

Next, for the additional CSE requirements from Definition 2:

1. Free Entry is satisfied because condition 2 ensures each $\tau \in T$ achieves value exactly equal to c;

2. Market Clearing follows directly from condition 4.

Thus, the tuple constitutes a CSE. ■

Proof of Lemma 4. This result follows directly from the incentive constraints in problem (7).

Choose any s-side type-i. If assume $\bar{U}^s(i) > 0$, the market for these agents clears in a CSE. That there is a equilibrium terms-of-trade that attracts them, with matching probability λ^s, fee ϕ^s, and partner distribution $G^{-s}(j)$. From problem (7):

$$\bar{U}^s(i) = \lambda^s \left(\int_{\mathbb{I}^{-s}} u^s(i,j) dG^{-s}(j) - \phi^s \right)$$

Conversely, all other types i' weakly prefer not to come to this market

$$\bar{U}^s(i') \geq \lambda^s \left(\int_{\mathbb{I}^{-s}} u^s(i',j) dG^{-s}(j) - \phi^s \right).$$

Subtracting inequalities gives us

$$\bar{U}^s(i') - \bar{U}^s(i) \geq \lambda^s \int_{\mathbb{I}^{-s}} \big(u^s(i',j) - u^s(i,j) \big) dG^{-s}(j).$$

Now we use the monotone WTP assumptions. With Increasing WTP, the right hand side of this last inequality is nonnegative when $i' > i$, so $\bar{U}^s$ is nondecreasing. With Decreasing WTP, the right hand side is nonnegative when $i' < i$, so $\bar{U}^s$ is nonincreasing.

Finally, if $\bar{U}^s(i) = 0$, nonnegativity of $\bar{U}^s$ implies $\bar{U}^s(i') \geq \bar{U}^s(i)$ for all i', ensuring weak monotonicity in this case as well. ∎

B Proof for Section 5

Proof of Lemma 5. From Lemma 4, Monotone Willingness-to-Pay implies that market utilities $\bar{U}^s(i)$ are monotone. By Lebesgue's differentiation theorem, any monotone function is differentiable almost everywhere.

Now fix a $\tau \in T$, with $\lambda^s = \Lambda^s(\tau)$ and G^s degenerate at k^s for $s \in \{a, b\}$. Also assume $k^a \in (0, 1)$, so that for all small positive ϵ, $0 \leq k^a - \epsilon < k^a + \epsilon \leq 1$. The constraints from problem (7) require

$$\begin{aligned}
\bar{U}^a(k^a + \epsilon) - \bar{U}^a(k^a) &\geq \lambda^a(u^a(k^a + \epsilon, k^b) - u^a(k^a, k^b)), \\
\bar{U}^a(k^a - \epsilon) - \bar{U}^a(k^a) &\geq \lambda^a(u^a(k^a - \epsilon, k^b) - u^a(k^a, k^b)),
\end{aligned}$$

or

$$\begin{aligned}
\frac{\bar{U}^a(k^a + \epsilon) - \bar{U}^a(k^a)}{\epsilon} &\geq \lambda^a \frac{(u^a(k^a + \epsilon, k^b) - u^a(k^a, k^b))}{\epsilon}, \\
\frac{\bar{U}^a(k^a) - \bar{U}^a(k^a - \epsilon)}{\epsilon} &\leq \lambda^a \frac{(u^a(k^a, k^b) - u^a(k^a - \epsilon, k^b))}{\epsilon}.
\end{aligned}$$

Since $\bar{U}^a$ is differentiable almost everywhere, it must be $\bar{U}^{a\prime}(k^a) = \lambda^a u_1^a(k^a, k^b)$ at such points.

A symmetric argument establishes the result for side b. ∎

Proof of Proposition 1. We seek to characterize an (i, i) market in a competitive search equilibrium in the symmetric case. If such an equilibrium exists, then for all $i = k^a = k^b$, the maximized value of Problem (9) must be c:

$$\begin{aligned}
c = \max_{\lambda \in [0, \bar{\lambda}]} \; & 2m(\lambda, \lambda)\left(u(i, i) - \frac{\bar{U}(i)}{\lambda}\right) \\
\text{s.t. } & \bar{U}(j) - \bar{U}(i) \geq \lambda(u(j, i) - u(i, i)) \text{ for all } j < i \\
& \bar{U}(i) \leq \lambda u(i, i)
\end{aligned} \tag{27}$$

We start by defining a threshold equilibrium utility $\hat{U}(i) \equiv \bar{\lambda} u(i,i)$. The last constraint of problem (28) requires that $\bar{U}(i) \le \lambda u(i,i) \le \bar{\lambda} u(i,i) = \hat{U}(i)$.

Unconstrained Markets. First suppose the constraints are all slack. Using equation (4), we can rewrite the unconstrained problem in terms of the number of agents per posting, $n = N(\lambda, \lambda)$:

$$c = \max_{n \ge 0} 2\big(M^*(n)u(i,i) - n\bar{U}(i)\big) \tag{28}$$

where $M^*(n) \equiv M(n,n)$. Then since M is strictly concave, the objective has a unique maximum satisfying $M^{*\prime}(n)u(i,i) = \bar{U}(i)$. Plugging this into the fact that the objective function must equal c, $c = 2\big(M^*(n)u(i,i) - n\bar{U}(i)\big)$, we get

$$\frac{c}{2u(i,i)} = M^*(n) - nM^{*\prime}(n)$$

Again using concavity of the matching function, the right hand side is increasing. It is equal to zero at $n = 0$ and, since $2u(i,i)m(0,0) > c$, it exceeds $c/(2u(i,i))$ as n grows without bound.[3] Thus n is uniquely defined if an unconstrained (i,i) market exists in equilibrium; call this value $n^*(i)$.

If $c > 0$, once we know $n = n^*(i)$, we can recover $\bar{U}(i) = \bar{U}^*(i)$ from the free entry condition: $c = 2\big(M^*(n^*(i))u(i,i) - n^*(i)\bar{U}^*(i)\big)$. We can also recover the matching probability $\ell^*(i) = L(n^*(i), n^*(i))$ from equation (3). Lastly, we can recover $\Phi^*(i)$ from $\bar{U}^*(i) = \ell^*(i)(u(i,i) - \Phi^*(i))$. Because $c > 0$, we must have $m(\ell(i), \ell(i)) \ge 0$ and positive fees, $\Phi^*(i) > 0$.

If $c = 0$, we get $n^*(i) = 0$ from the described problem. In this case $\ell^*(i) = \bar{\lambda}$. Since $M^*(0) = 0$, we cannot use the logic in the $c > 0$ case to recover $\bar{U}^*(i)$ and $\Phi^*(i)$. When $\bar{U}(i) < \hat{U}(i)$, the maximized value of the right-hand side of equation (28) delivers a positive value, which cannot be the solution. When $\bar{U}(i) = \hat{U}(i)$, the maximized value of the right-hand side of equation (28) delivers a zero value, which can be a solution. Using the definition of $\hat{U}(i)$ and the last constraint of problem (27), we have ruled out any $\bar{U}(i) > \hat{U}(i)$. Thus if market (i,i) is active in the competitive search equilibrium, it must be $\bar{U}^*(i) = \hat{U}(i)$. Correspondingly, the matching probability is $\bar{\lambda}$, and the fee charged is $\Phi^*(i) = u(i,i) - \bar{U}^*(i)/\bar{\lambda} = 0$.

If there is a competitive search equilibrium with PAM, this determines the matching probability and agent value when $i = 0$, since there are no incentive constraints in the $(0,0)$ market.

[3] It is straightforward to prove from the definitions of M^*, m, n, and λ that $\lim_{n\to\infty} M^*(n) - nM^{*\prime}(n) = m(0,0)$.

Constrained Markets. Now suppose one or more of the constraints in problem (27) is binding. Since the objective function is decreasing in $\bar{U}(i)$, if $\bar{U}(i) > \bar{U}^*(i)$ the objective evaluates to less than c for all values of λ. This is inconsistent with an active (i,i) market in a competitive search equilibrium. We thus conclude that $\bar{U}(i) \leq \bar{U}^*(i)$ in any competitive search equilibrium.

For $\bar{U}(i) < \bar{U}^*(i)$, there are two values of n that satisfy the free entry condition

$$c = 2\big(M^*(n)u(i,i) - n\bar{U}(i)\big).$$

This is because (i) the right hand side is zero when $n = 0$; (ii) the right hand side is negative for a sufficiently large value of n since the slope of $M^*(n)$ is less than 1 and decreasing for all positive n; (iii) the right hand side is strictly increasing and weakly larger than c when $n = n^*(i)$ since $\bar{U}(i) < \bar{U}^*(i)$;[4] and (iv) the right hand side is a strictly concave function of n. This implies that there are two solutions to this equation, one in the interval $[0, n^*(i)]$ and the other in $(n^*(i), \infty)$.

Now since $M^*(n)$ is concave and $M^*(n) \leq n$ implies $M^*(0) = 0$, agents' matching rate $M^*(n)/n$ is a decreasing function of n. Thus we can equivalently state that for $\bar{U}(i) < \bar{U}^*(i)$, there are two solutions $\lambda \in [0, \bar{\lambda}]$ to the equation

$$c = 2m(\lambda, \lambda)\left(u(i,i) - \frac{\bar{U}(i)}{\lambda}\right).$$

One of the solutions is in the interval $[\ell^*(i), \bar{\lambda}]$, while the other lies in the interval $(0, \ell^*(i))$. Which solution is relevant depends on whether we have Increasing or Decreasing WTP, and so we treat those cases separately.

Increasing WTP (IWTP). First, suppose there is IWTP. The constraints in Problem (27), which only hold for $j < i$, imply a lower bound on λ:

$$\lambda \geq \sup_{j<i} \frac{\bar{U}(i) - \bar{U}(j)}{u(i,i) - u(j,i)}.$$

Now if the solution to problem (27) has $\ell(i)$ equal to the smaller solution, i.e. $\ell(i) \in (0, \ell^*(i))$, then $\ell^*(i)$ is feasible and, since $\bar{U}(i) < \bar{U}^*(i)$, attains a value in problem (27) that exceeds c, a contradiction. So it must be that the solution to problem (27) has $\ell(i)$ equal to the larger solution, $\ell(i) \in [\ell^*(i), \bar{\lambda}]$, whenever $\bar{U}(i) < \bar{U}^*(i)$ in a competitive search equilibrium in the IWTP case.

[4] If $c = 0$, then $n^*(i) = 0$ and the equation is satisfied at $n = 0$ even if $\bar{U}(i) < \bar{U}^*(i)$.

Next, we replace the global incentive constraints in Problem (27) with local ones:

$$
\begin{aligned}
c = &\max_{\lambda \in [\ell^*(i), \bar{\lambda}]} 2m(\lambda, \lambda) \left(u(i,i) - \frac{\bar{U}(i)}{\lambda} \right) \\
&\text{s.t. } \bar{U}'(i) = \lambda u_1(i,i).
\end{aligned}
$$

There is nothing to optimize here. Instead, these equations define the path of $\bar{U}(i)$ and $\ell(i)$:

$$
c = 2m(\ell(i), \ell(i)) \left(u(i,i) - \frac{\bar{U}(i)}{\ell(i)} \right), \quad \bar{U}'(i) = \ell(i) u_1(i,i), \quad \text{and } \ell(i) \in [\ell^*(i), \bar{\lambda}]. \tag{29}
$$

Since $\bar{U}(i)$ is continuous, the first equation and the restriction that $\ell(i) \in [\ell^*(i), \bar{\lambda}]$ ensures $\ell(i)$ is continuous. Therefore from the second equation $\bar{U}'(i)$ is continuous.

Now solve the first equation in (29) for $\bar{U}(i)$, differentiate with respect to i, and eliminate $\bar{U}'(i)$ using the second equation. This gives us equation (13) in the text and establishes that it must hold at all i, since $\ell(i)$ is continuous and $\bar{U}(i)$ is continuously differentiable.

We next prove monotonicity. We know $\bar{U}(i)$ is increasing from the local incentive constraint $\bar{U}'(i) = \ell(i)u_1(i,i)$. Because $2u(0,0)m(0,0) > c$, $\bar{U}(0) > 0$, and monotonicity of $\bar{U}(i)$ implies that $\bar{U}(i) > 0$ for $i > 0$. To prove $\ell(i)$ is increasing, we first establish that for fixed i, h in equation (12) is single-peaked in $\lambda \in [0, \bar{\lambda}]$, achieving maximum value $\bar{U}^*(i)$ at $\ell(i) = \ell^*(i)$, the unconstrained case. To prove this, note that we have already established that the objective function in problem (27) is single-peaked: for all $\bar{U}(i) < \bar{U}^*(i)$, there exists thresholds λ_1 and λ_2 with $\lambda_1 < \ell^*(i) \leq \lambda_2$ such that

$$
c < 2m(\lambda, \lambda) \left(u(i,i) - \frac{\bar{U}(i)}{\lambda} \right) \Leftrightarrow \lambda \in (\lambda_1, \lambda_2).
$$

Flipping this inequality around, we get that

$$
\bar{U}(i) < \lambda \left(u(i,i) - \frac{c}{2m(\lambda, \lambda)} \right) = h(\lambda, i) \Leftrightarrow \lambda \in (\lambda_1, \lambda_2).
$$

This proves that $h(\lambda, i)$ is single-peaked. Moreover, setting $\bar{U}(i) = \bar{U}^*(i)$ and using a similar logic implies that $\max_{\lambda \in [0, \bar{\lambda}]} h(\lambda, i) = h(\ell^*(i), i) = \bar{U}^*(i)$.

When $c > 0$, since private information leads to $\ell(i) \geq \ell^*(i)$ in the IWTP case, this means $h_1(\ell(i), i) < 0$. Then the differential equation (13) implies $\ell'(i) > 0$, proving ℓ is increasing. When $c = 0$, $\ell^*(i) = \bar{\ell}$, so $\ell(i) \in [\ell^*(i), \bar{\lambda}]$ implies $\ell(i) = \bar{\lambda}$, weakly increasing as well. Finally, to show that $\Phi(i)$ is increasing, we totally differentiate the constraint

$\bar{U}(i) = \lambda(i)(u(i,i) - \Phi(i))$:

$$\bar{U}'(i) = \lambda(i)(u_1(i,i) + u_2(i,i) - \Phi'(i)) + \lambda'(i)(u(i,i) - \Phi(i)).$$

The local incentive constraint requires that $\bar{U}'(i) = \lambda(i)u_1(i,i)$, so we can write:

$$\Phi'(i) = u_2(i,i) + \frac{\lambda'(i)}{\lambda(i)^2}\bar{U}(i)$$

The right-hand side is positive because of common ranking, $\lambda'(i) \geq 0$, and $\bar{U}(i) > 0$. Thus $\Phi(i)$ must be increasing as well.

Decreasing WTP (DWTP). With DWTP, the logic is reversed. The constraints in Problem (27), specialized to the symmetric case imply an upper bound the meeting rate:

$$\lambda \leq \sup_{j<i} \frac{\bar{U}(j) - \bar{U}(i)}{u(j,i) - u(i,i)}.$$

We can use that to rule out the upper solution, which is dominated by $\ell^*(i)$. We thus conclude that the solution to problem (27) has $\ell(i)$ equal to the smaller solution, $\ell(i) \in [0, \ell^*(i))$, whenever $\bar{U}(i) < \bar{U}^*(i)$ in a competitive search equilibrium in the DWTP case.

We again use the local incentive constraints to get a version of equation (29):

$$c = 2m(\ell(i), \ell(i)) \left(u(i,i) - \frac{\bar{U}(i)}{\ell(i)}\right), \quad \bar{U}'(i) = \ell(i)u_1(i,i), \quad \text{and } \ell(i) \in [0, \ell^*(i)). \tag{30}$$

Again, we use continuity of $\bar{U}(i)$ to prove continuity of $\ell(i)$ and then use that to prove continuity of $\bar{U}'(i)$. Finally, the same algebraic manipulations give us equation (13) in the text and establishes that it must hold at all i.

The monotonicity proof is also reversed. We know $\bar{U}(i)$ is decreasing from the local incentive constraint $\bar{U}'(i) = \ell(i)u_1(i,i)$. To prove $\ell(i)$ is decreasing, we use the same property of h in equation (12). Now private information leads to $\ell(i) < \ell(i^*)$ in the DWTP case, implying $h_1(\ell(i), i) < 0$. Then the differential equation (13) implies $\ell'(i) < 0$, proving ℓ is decreasing.

Finally, since $\ell(i)$ is decreasing, for the $c > 0$ case, $m(\ell(i), \ell(i))$ is increasing and so the free-entry condition $c = 2m(\ell(i), \ell(i))\Phi(i)$ implies $\Phi(i)$ is decreasing. When $c = 0$, we have shown that $\ell(i) < \ell^*(i)$ for $i > 0$ and hence $m(\ell(i), \ell(i)) > m(\bar{\lambda}, \bar{\lambda}) = 0$. Thus the free-entry condition implies $\Phi(i) = 0$, which is (weakly) increasing as well. ■

Proof of Proposition 2.

Characterizing the Markets. We start by characterizing the (i, i) market with no information friction and $c = 0$, as in equation (12). For any i, the solution to problem (12) is $\ell^*(i) = \bar{\lambda}$, as the marginal benefit of increasing the matching probability is always positive, and the marginal cost is zero. We have shown that the terms-of-trade that attract type $(0, 0)$ terms-of-trade must be unconstrained, so $\ell(0) = \ell^*(0) = \bar{\lambda}$. The equilibrium utility and the fee can be accordingly characterized, with $\bar{U}(0) = \bar{\lambda}u(0, 0)$ and $\phi(0) = 0$.

We also showed that under increasing WTP, $L(i)$ is increasing in i, $\bar{\lambda} = \ell(0) \leq \ell(i) \leq \bar{\lambda}$, $\forall i$. Thus, it must be that $\ell(i) = \bar{\lambda}$. The matching probability for each unit of platform effort is $m(\bar{\lambda}, \bar{\lambda}) = 0$. Thus, the free-entry condition does not impose any restrictions on the fees. Instead, we have shown that local incentive constraints bind, $\bar{U}'(i) = \bar{\lambda}u_1(i, i)$, and the equilibrium utility also satisfies $\bar{U}(i) = \bar{\lambda}(u(i, i) - \Phi(i))$. Totally differentiate this constraint we obtain: $\bar{U}'(i) = \bar{\lambda}(\partial_1 u(i, i) + \partial_2 u(i, i) - \Phi'(i))$. Since $\bar{U}'(i) = \bar{\lambda}u_1(i, i)$, it must be that $\Phi'(i) = \partial_2 u(i, i)$. This completes the characterization of the equilibrium outcomes.

Existence of PAM CSE. We conjecture that there is a PAM CSE and look at a market that attracts type i agents on side a and type j agents on side b and $i \neq j$. The condition $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}\right) \notin \mathbb{\Lambda}$ rules out these deviations by making the matching probabilities that are consistent with the local ICs infeasible under the matching function, and, thus, there is a PAM CSE. WLOG, assume $j > i > 0$. In the conjectured market, the local incentive constraints require

$$\bar{U}'(i) = \lambda^a u_1(i, j),\ \bar{U}'(j) = \lambda^b u_1(j, i).$$

From the characterization of the PAM equilibrium, we have already shown that $\bar{U}'(i) = \bar{\lambda}u_1(i, i)$ and $\bar{U}'(j) = \bar{\lambda}u_1(j, j)$. Combine it with the incentive constraints for the (i, j) terms-of trade, we have two equations for (λ^a, λ^b):

$$\lambda^a = \bar{\lambda}\frac{u_1(i, i)}{u_1(i, j)},\ \ \lambda^b = \bar{\lambda}\frac{u_1(j, j)}{u_1(j, i)}.$$

Because $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}\right) \notin \mathbb{\Lambda}$, any combination of such matching probabilities is infeasible.

The argument so far imposes ICs on both sides of the market, which does not applies to a $(0, j)$ market because such a market does not have an IC on side-a. Under condition (b), the IC-b in such a $(0, j)$ market requires for any $k < j$:

$$\bar{U}(k) - \bar{U}(j) \geq \lambda^b(u(k, 0) - u(1, 0)) = 0.$$

We have shown that $\bar{U}'(i) = \bar{\lambda} u_1(i, i) \geq 0$. The IC-$b$ can be satisfied only when for any $k < j$, $\bar{U}(k) = \bar{U}(0)$. Creating a (j, j) market with (λ^a, λ^b) will be strictly more profitable and feasible. This rules out a $(0, j)$ market.

■

Proof of Proposition 3. We follow the structure of the proof of Proposition 2.

Characterization. Assume a competitive search equilibrium exists with PAM. With Common Ranking, Supermodularity, and Decreasing WTP, the same logic as in the proof of Proposition 2 implies that: $\ell(0) = \bar{\lambda}$, $\Phi(0) = 0$, $\bar{U}(0) = \bar{\lambda} u(0, 0)$.

With decreasing WTP, Proposition 1 proves that $\Phi(i)$ is decreasing in i. If we can rule out the possibilities of $\Phi(i) < 0$ for $i > 0$, then we can establish that $\Phi(i) = 0$. For $\Phi(i) < 0$ in the equilibrium, it must $\ell(i) = \bar{\lambda}$. Suppose this is indeed the case, type 0 would be strictly better off deviating to such terms-of-trade. More precisely, by participating to such a (i, i) market, the type 0 receives a payoff of:

$$\bar{\lambda}(u(0, i) - \Phi(i)) > \bar{\lambda} u(0, i) > \bar{\lambda} u(0, 0) = \bar{U}(0),$$

where the first inequality uses the assumption $\Phi(i) > 0$, and the second inequality uses common ranking. This violates the IC. Thus, for $i > 0$, $\Phi(i) = 0$. Next, zero fees imply that $\bar{U}(i) = \ell(i) u(i, i)$. Totally differentiating this condition, we have $\bar{U}'(i) = \ell'(i) u(i, i) + \ell(i)(\partial_1 u(i, i) + \partial_2 u(i, i))$. Imposing the IC $\bar{U}'(i) = \ell(i) u_1(i, i)$, we have $\ell'(i) u(i, i) = -\ell(i) u_2(i, i)$. This differential equation, with the initial condition $\ell(0) = \bar{\lambda}$, has a solution:

$$\ell(i) = \bar{\lambda} \exp\left(-\int_0^i \frac{u_2(\iota, \iota)}{u(\iota, \iota)} d\iota\right) > 0.$$

Thus we can divide the condition $\bar{U}'(i) = \bar{\lambda} u_1(i, i)$ by $\bar{U}(i) = \bar{\lambda} u(i, i)$ to get $\frac{\bar{U}'(i)}{\bar{U}(i)} = \frac{u_1(i,i)}{u(i,i)}$.

Sufficient Conditions We look for sufficient conditions for a PAM CSE. We conjecture that there is a PAM CSE and look at a market which attracts type i agents on side a and type j agents on side b. If all such markets are unprofitable, then there is a PAM CSE. To start, we assume that both $i > 0$ and $j > 0$.

In the conjectured market, the local incentive constraints require

$$\lambda^a = \frac{\bar{U}'(i)}{u_1(i,j)},\ \lambda^b = \frac{\bar{U}'(j)}{u_1(j,i)}.$$

Inverting the incentive constraints for type i and type j, we get:

$$\begin{aligned}\phi^a + \phi^b &= u(i,j) + u(j,i) - \frac{\bar{U}(i)}{\bar{U}'(i)} u_1(i,j) - \frac{\bar{U}(j)}{\bar{U}'(j)} u_1(j,i) \\ &= u(i,j) + u(j,i) - u(i,i)\frac{u_1(i,j)}{u_1(i,i)} - u(j,j)\frac{u_1(j,i)}{u_1(j,j)},\end{aligned}$$

where in the second equality we used the solution from PAM CSE. Under assumption 6, $\phi^a + \phi^b < 0$. Thus there is no terms-of-trade attracting (i,j) and delivers non-negative payoffs to platforms.

Now consider a terms-of-trade attracting $(0,j)$, where $j > 0$. In this case, the sum of the fees is

$$\phi^a + \phi^b = u(0,j) + u(j,0) - u(0,0)\frac{\bar{\lambda}}{\lambda^a} - u(j,j)\frac{u_1(j,0)}{u_1(j,j)}.$$

Because $\lambda^a \leq 1$,

$$\phi^a + \phi^b \leq u(0,j) + u(j,0) - u(0,0)\bar{\lambda} - u(j,j)\frac{u_1(j,0)}{u_1(j,j)}.$$

Under assumption 6, $\phi^a + \phi^b \leq 0$. ■

C Proof for Section 6

C.1 Proof of Lemma 7

We start by defining the objective function of the platforms in terms of (n^a, n^b): $W(n^a, n^b) \equiv M(n^a, n^b)\big(u^a(0,0) + u^b(0,0)\big) - \bar{U}^a(0)n^a - \bar{U}^b(0)n^b$. As M is strictly concave, there is a unique maximizer to $W(n^a, n^b)$ given any $\bar{U}^a(0)$ and $\bar{U}^b(0)$. Correspondingly, there is a unique combination $(\lambda^a(0), \lambda^b(0))$. In a PAM CSE, the market clearing condition requires that $\lim_{i\to 0^+}\sigma(i) = 0$. Otherwise, there is $j \in \mathbb{I}^b$ whose market does not clear.

Now we prove the continuity of $\lambda^s(i)$ by contradition. Suppose, to the contrary of the lemma, $\lambda^s(i)$ is not continuous at 0. Let $l^s = \lim_{i\to 0^+}\lambda^s(i)$. For $i > 0$,

$$c = \hat{V}(i,\sigma(i)) = m(\lambda^a(i), \lambda^b(i))\left(u(i,\sigma(i)) + u^b(\sigma(i), i) - \frac{\bar{U}^a(i)}{\lambda^a(i)} - \frac{\bar{U}^b(\sigma(i))}{\lambda^b(i)}\right)$$

Taking $i \to_+ 0$:

$$c = \lim_{i \to_+ 0} \hat{V}(i, \sigma(i)) = m(l_0^a, l_0^b) \left(u^a(0,0) + u^b(0,0) - \frac{\bar{U}^a(0)}{l_0^a} - \frac{\bar{U}^b(0)}{l_0^b} \right).$$

If $(l^a, l^b) \neq (\lambda^a(0), \lambda^b(0))$, we have found another maximiser for the platform payoff at $(0,0)$. A contradiction.

C.2 Proof of Lemma 8

It is convenient to use the following notations, where we write the payoff of platforms as a function of $(\lambda^s, k^s)_{s=a,b}$

$$\tilde{V}(\lambda^a, \lambda^b, k^a, k^b) \equiv m(\lambda^a, \lambda^b) \sum_{s=a,b} \left(u^s(k^s, k^{-s}) - \frac{\bar{U}^s(k^s)}{\lambda^s} \right),$$

and

$$\hat{\lambda}^s(k^s, k^{-s}) = \frac{\bar{U}^{s\prime}(k^s)}{u_1^s(k^s, k^{-s})}, \text{ for } s = a, b. \tag{31}$$

With this notation:

$$\hat{V}(i,j) = \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j).$$

Among the odes in the lemma, equation (21) comes directly from the local incentive constraints in lemma 5. We start by deriving equation (20).

For a fixed $i \in \mathbb{I}^a$, $\sigma(i)$ has to be the local maximum of $\hat{V}(i,j)$. Otherwise, the platforms can increase their payoffs by choosing $j' \neq \sigma(i)$, violating the optimality condition. Thus, the first-order condition holds $\hat{V}_2(i, \sigma(i)) = 0$. The same argument implies $\hat{V}_1(i, \sigma(i)) = 0$.

Given j, the impact of changing i on $\hat{V}(i,j)$ can be written in terms of three effects: the change in matching probabilities on two sides and the change in the utility from matches:

$$\begin{aligned}
&\frac{\partial \log \hat{V}(i,j)}{\partial i} \\
=&\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial \hat{\lambda}^a(i,j)} \hat{\lambda}_1^a(i,j) + \frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial \hat{\lambda}^b(i,j)} \hat{\lambda}_2^b(j,i) && (32) \\
+&\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial i}. && (33)
\end{aligned}$$

We now unpack the three effects in turns. Starting from the effect through matching probability λ^a. From the definition of $\tilde{V}$:

$$\frac{\partial \log \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}{\partial \lambda^a} = \left(\frac{\epsilon_1(\lambda^a, \lambda^b)}{m(\lambda^a, \lambda^b)} \tilde{V}(\lambda^a, \lambda^b, k^a, k^b) + \frac{\bar{U}^a(k^a)}{\lambda^a} \right) \frac{m(\lambda^a, \lambda^b)}{\lambda^a \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}.$$

In the PAM CSE, $\tilde{V}(\lambda^a(i), \lambda^b(i), i, \sigma(i)) = c$, and so we can write

$$\frac{\partial \log \tilde{V}(\lambda^a(i), \lambda^b(i), i, \sigma(i))}{\partial \lambda^a(i)} = \nu^a(i) \frac{m(\lambda^a(i), \lambda^b(i))}{c\lambda^a(i)},$$

where we used the definition of $\nu^a(i)$ as in the lemma. The same logic establishes that $\frac{\partial \log \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}{\partial \lambda^b} = \nu^b(i) \frac{m(\lambda^a(i), \lambda^b(i))}{c\lambda^b(i)}$. To derive $\hat{\lambda}^s_2(i,j)$, we totally differentiate the identity $\lambda^a(i) = \hat{\lambda}^a(i, \sigma(i))$ and $\lambda^b(i) = \hat{\lambda}^b(\sigma(i), i)$ to get:

$$\lambda^{a\prime}(i) = \hat{\lambda}^a_1(i, \sigma(i)) + \hat{\lambda}^a_2(i, \sigma(i))\sigma'(i)$$

$$\lambda^{b\prime}(i) = \hat{\lambda}^b_1(\sigma(i), i)\sigma'(i) + \hat{\lambda}^b_2(\sigma(i), i).$$

Differentiating equation (31), we have:

$$\hat{\lambda}^a_2(i, \sigma(i)) = -\lambda^a(i) \frac{u^a_{1,2}(i, \sigma(i))}{u^a_1(i, \sigma(i))},\ \hat{\lambda}^b_2(\sigma(i), i) = -\lambda^b(i) \frac{u^b_{1,2}(\sigma(i), i)}{u^b_1(\sigma(i), i)}.$$

So

$$\hat{\lambda}^a_1(i, \sigma(i)) = \lambda^{a\prime}(i) + \sigma'(i)\lambda^a(i) \frac{u^a_{1,2}(i, \sigma(i))}{u^a_1(i, \sigma(i))},$$

$$\hat{\lambda}^b_1(\sigma(i), i) = \frac{1}{\sigma'(i)} \left(\lambda^{b\prime}(i) + \lambda^b(i) \frac{u^b_{1,2}(\sigma(i), i)}{u^b_1(\sigma(i), i)} \right).$$

Putting this together, we have

$$\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial \hat{\lambda}^a(i,j)} \hat{\lambda}^a_1(i,j) = \nu^a(i) \frac{m(\lambda^a(i), \lambda^b(i))}{c} \left(\frac{\lambda^{a\prime}(i)}{\lambda^a(i)} + \sigma'(i) \frac{u^a_{1,2}(i, \sigma(i))}{u^a_1(i, \sigma(i))} \right),$$

$$\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial \hat{\lambda}^b(i,j)} \hat{\lambda}^b_2(j,i) = \nu^b(i) \frac{m(\lambda^a(i), \lambda^b(i))}{c} \left(-\frac{u^b_{1,2}(\sigma(i), i)}{u^b_1(\sigma(i), i)} \right).$$

Finally, we directly compute the effect of changing matching partners on utility:

$$\frac{\partial \log \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}{\partial i} = \frac{m(\lambda^a, \lambda^b)}{\tilde{V}(\lambda^a, \lambda^b, k^a, k^b)} \left(u^a_1(k^a, k^b) + u^b_2(k^b, k^a) - \frac{\bar{U}^{a\prime}(k^a)}{\lambda^a} \right).$$

The incentive constraint (31) and the fact that in the PAM CSE, $\tilde{V}(\lambda^a(i), \lambda^b(i), i, \sigma(i)) = c$

means that we can write this as

$$\frac{\partial \log \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}{\partial i} = \frac{m(\lambda^a, \lambda^b)}{c} u_2^b(k^b, k^a).$$

Now we have all terms needed to calculate $\frac{\partial \hat{V}(i,j)}{\partial i}$. Setting $\frac{\partial \hat{V}(i,j)}{\partial i} = 0$ and multiplying both sides by $\frac{c}{m(\lambda^a(i), \lambda^b(i))}$, we get the following equation:

$$0 = \nu^a(i) \left(\frac{\lambda^{a'}(i)}{\lambda^a(i)} + \sigma'(i) \frac{u_{1,2}^a(i, \sigma(i))}{u_1^a(i, \sigma(i))} \right) - \nu^b(i) \frac{u_{1,2}^b(\sigma(i), i)}{u_1^b(\sigma(i), i)} + u_2^b(\sigma(i), i).$$

Similarly, for the b side:

$$0 = -\nu^a(i) \frac{u_{1,2}^a(i, \sigma(i))}{u_1^a(i, \sigma(i))} + \nu^b(i) \frac{1}{\sigma'(i)} \left(\frac{\lambda^{b'}(i)}{\lambda^b(i)} + \frac{u_{1,2}^b(\sigma(i), i)}{u_1^b(\sigma(i), i)} \right) + u_2^a(i, \sigma(i)).$$

Using the definition of $\iota(i)$, we reach the formula as in the lemma.

Now we move on to equation (22). In our model, matches are one-to-one. Thus, $\forall i \in \mathbb{I}^a$, the accumulated numbers of matches on the a side must equal the accumulated numbers of matches on the b side:

$$\int_0^i \lambda^a(i') \bar{g}^a(i') di' = \int_0^{\sigma(i)} \lambda^b(\sigma^{-1}(j')) \bar{g}^b(j') dj'.$$

The equation above has to hold for any i, we differentiate both sides with respect to i:

$$\lambda^a(i) \bar{g}^a(i) = \lambda^b(i) \bar{g}^b(\sigma(i)) \sigma'(i).$$

With the assumption that $\bar{g}^b(j) > 0$ for all $j \in \mathbb{I}^b$, we derive the ode for $\sigma(i)$:

$$\sigma'(i) = \frac{\lambda^a(i)}{\lambda^b(i)} \frac{I^a \bar{g}^a(i)}{I^b \bar{g}^b(\sigma(i))}.$$

For the market clearing condition to hold for all types, we further require that $\sigma(0) = 0$ and $\sigma(1) = 1$. This conclude the proof.

C.3 Details of Model with Observable Types

A market is a vector $\tau = (\boldsymbol{\phi}^s, G^s)_{s=a,b}$ satisfying $\boldsymbol{\phi}^s \equiv (\phi_1^s, \ldots, \phi_I^s) \in \mathbb{R}^I$ and $G^s \equiv (G_1^s, \ldots, G_I^s) \in \Delta^I$, the standard probability simplex. We interpret ϕ_i^s to be the fee paid by a type i agent on side s of the market when he matches; and G_i^s to be the share of type i

agents on side s of the market. Let $\mathbb{M}_0$ denote the set of markets, i.e. the set of vectors satisfying these conditions. With this definition of markets, the value of platforms and agents are accordingly

$$\tilde{V}(\tau) \equiv m\left(N^a(\tau), N^b(\tau)\right)\left(\sum_{i=1}^{I} \phi_i^a \omega_i^a + \sum_{j=1}^{I} \phi_j^b \omega_j^b\right), \tag{34}$$

and

$$\tilde{U}_i^s(\tau) \equiv \frac{1}{N^s(\tau)} m\left(N^a(\tau), N^b(\tau)\right)\left(\sum_{j=1}^{I} G_j^{-s} u_{i,j} - \phi_i^s\right), \tag{35}$$

with

$$\tilde{U}_i^s(\tau) \equiv \begin{cases} \infty \\ 0 & \text{if } \sum_{j=1}^{I} G_j^{-s} u_{i,j} \gtreqless \phi_i^s \text{ and } N^s(\tau) = 0. \\ -\infty \end{cases} \tag{36}$$

The logic closely follows equations (2)–(**??**), extended to allow for type-specific fees.

Definition 5 *A **partial equilibrium with observable types** $\{N, M^p, M, \bar{\boldsymbol{U}}\}$ is a mapping $N : \{a, b\} \times \mathbb{M}_0 \to \mathbb{R}_+$, two nonempty sets $M \subseteq M^p \subseteq \mathbb{M}_0$, and strictly positive numbers $\bar{\boldsymbol{U}} \equiv \bar{U}_1, \ldots, \bar{U}_I$ such that:*

1. *(Optimal Search) $\forall m \in \mathbb{M}_0$, $s \in \{a, b\}$, and $i \in \mathbb{I}$, $\bar{U}_i \geq \tilde{U}_i^s(m)$; and if $N^s(m) > 0$, $\bar{U}_i = \tilde{U}_i^s(m)$ for some $i \in \mathbb{I}$;*
2. *(Promise Keeping) $M^p = \{m \in \mathbb{M}_0 | G_i^s > 0 \Rightarrow \bar{U}_i = \tilde{U}_i^s(m)\}$;*
3. *(Profit Maximization) $M = \arg\max_{m \in M^p} \tilde{V}(m)$.*

With these definition of partial equilibrium, the definition of a competitive search equilibrium with observable types is:

Definition 6 *A **competitive search equilibrium with observable types** is a partial equilibrium $\{N, M^p, M, \bar{\boldsymbol{U}}\}$ and a measure μ on the set of active market M such that*
1. (free entry) $c = V(m)$ for all $m \in M$;
2. (market clearing) $\bar{\omega}_i = \int_M \left[\sum_{s=a,b} G_i^s N^s(m)\right] d\mu(m)$.

We start by showing that the outcomes of a competitive search equilibrium can be characterized by a set of optimization problem. Consider the following problem given

$\bar{\boldsymbol{U}} \equiv U_1, \ldots, U_I$:

$$\bar{V} = \max_{m \in \mathbb{M}} (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \sum_{s=a,b} \sum_{i=1}^{I} \phi_i^s G_i^s, \tag{37}$$

$$\text{s.t. } \bar{U}_i \geq \lambda^s \left(\sum_{j=1}^{I} \omega_j^{-s} u_{i,j} - \phi_i^s \right) \text{ with equality if } G_i^s > 0,$$

$$\text{for } s = a, b \text{ and } i \in \{1, \ldots, I\}$$

We say that $(M, \bar{V})$ solves optimization problem (37) if $\bar{V}$ is the maximum value of the optimization problem (37) and any $m \in M$ achieves this value given $\bar{\boldsymbol{U}}$.

Lemma 11 *Suppose $\{M, \bar{V}, \bar{\boldsymbol{U}}\}$ is a partial equilibrium with observable types. Then $(M, \bar{V})$ must solve problem (37) given $\bar{\boldsymbol{U}}$. Conversely, if $(M, \bar{V})$ solves optimization problem (37) given $\bar{\boldsymbol{U}}$, then $\{M, \bar{V}, \bar{\boldsymbol{U}}\}$ is a partial equilibrium with observable types.*

Proof. First, suppose $\{M, \bar{V}, \bar{\boldsymbol{U}}\}$ is a partial equilibrium, yet $(M, \bar{V})$ is not a solution to (37). Thus there is $m' \in \mathbb{M}$ such that the constraint of (37) is satisfied and $V(m') > \bar{V}$. This contradicts point 2 of the definition for a partial equilibrium.

Conversely, suppose $(M, \bar{V})$ is the solution to (37) given $\bar{\boldsymbol{U}}$ yet $(M, \bar{V}, \bar{\boldsymbol{U}})$ is not a partial equilibrium. If $\exists m \in M$ such that $V(m) \neq \bar{V}$. This violates that M is the solution to (37) because either m is not a maximizer or $\bar{V}$ is not the optimal value; If there is $m \in \mathbb{M}$ that violates condition 2 of the definition for a partial equilibrium, this m satisfies constraints of (37), and $V(m) > \bar{V}$, which contradicts $(M, \bar{V})$ solving (37). ■

We first restrict attention to separating competitive search equilibrium, then show that under monotonicity of $f_{i,j}$ in j, any competitive search equilibrium must be separating. With separation, the problem (37) can be analyzed in two steps: (1) conditional on a fixed pair (i, j), find the $(\boldsymbol{\lambda}, \boldsymbol{\phi})$ that maximizes the platform's value. (2) choose (i, j) that delivers the highest value to platforms. We define this step-one problem as

$$V_{i,j} = \max_{\boldsymbol{\lambda} > 0, \boldsymbol{\phi} > 0} m(\lambda^a, \lambda^b) \left(\phi^a + \phi^b \right), \tag{38}$$

s.t.

$$\bar{U}_i = \lambda^a \left(u_{i,j} - \phi^a \right),$$

$$\bar{U}_j = \lambda^b \left(u_{j,i} - \phi^b \right).$$

With observable types, the relevant value for the solution of problem 38 is the joint surplus in a (i, j) match, which we define as $f_{ij} \equiv u_{i,j} + u_{j,i}$.

Lemma 12 *Given (i, j), a unique solution to (38) exists:*

$$\lambda_{i,j}^a = \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j}},\ \lambda_{i,j}^b = \frac{\bar{U}_j}{\frac{1-\gamma}{2} f_{i,j}},\ \phi_{i,j}^a = u_{i,j} - \frac{1-\gamma}{2} f_{i,j},\ \phi_{i,j}^b = u_{j,i} - \frac{1-\gamma}{2} f_{i,j},$$

$$V_{i,j} = \gamma \left(\frac{1-\gamma}{2}\right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i \bar{U}_j\right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}}.$$

Proof. By definition:

$$V_{i,j} = \max_{\lambda^a, \lambda^b} v(\lambda^a, \lambda^b) = \max_{\lambda^a, \lambda^b} m(\lambda^a, \lambda^b) \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right).$$

The partial derivatives of $V_{i,j}$ are:

$$\frac{\partial \log v}{\partial \lambda^s}(\lambda^a, \lambda^b) = \frac{\partial_s m(\lambda^a, \lambda^b)}{m(\lambda^a, \lambda^b)} + \frac{\bar{U}_i}{\lambda^{s2}} \frac{1}{f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b}}.$$

For $\lambda^a, \lambda^b > 0$ and $f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} > 0$, this implies that

$$\frac{\partial \log v}{\partial \lambda^a}(\lambda^a, \lambda^b) \gtreqless 0 \Leftrightarrow (1 - \frac{1-\gamma}{2}) \frac{\bar{U}_i}{\lambda^a} + \frac{1-\gamma}{2} \frac{\bar{U}_j}{\lambda^b} \gtrless \frac{1-\gamma}{2} f_{i,j}$$
$$\frac{\partial \log v}{\partial \lambda^b}(\lambda^a, \lambda^b) \gtreqless 0 \Leftrightarrow (1 - \frac{1-\gamma}{2}) \frac{\bar{U}_j}{\lambda^b} + \frac{1-\gamma}{2} \frac{\bar{U}_i}{\lambda^a} \gtrless \frac{1-\gamma}{2} f_{i,j}$$

Thus when λ^a and λ^b are both close to 0, raising either of them increases v. When both are sufficiently large, reducing either of them increases v. If $\frac{\bar{U}_i}{\lambda^a} > \frac{\bar{U}_j}{\lambda^b}$, $0 < \gamma < 1$ implies $(1 - \frac{1-\gamma}{2}) \frac{\bar{U}_i}{\lambda^a} + \frac{1-\gamma}{2} \frac{\bar{U}_j}{\lambda^b} > (1 - \frac{1-\gamma}{2}) \frac{\bar{U}_j}{\lambda^b} + \frac{1-\gamma}{2} \frac{\bar{U}_i}{\lambda^a}$, and so we can have $\frac{\partial v}{\partial \lambda^a}(\lambda^a, \lambda^b) > 0 > \frac{\partial v}{\partial \lambda^b}(\lambda^a, \lambda^b)$, i.e. we increase v by increasing λ^a and decreasing λ^b. We can reach the opposite conclusion when $\frac{\bar{U}_i}{\lambda^a} < \frac{\bar{U}_j}{\lambda^b}$. This implies that v is single-peaked in $\boldsymbol{\lambda}$ such that $\lambda^a, \lambda^b > 0$ and $f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} > 0$. That peak comes at $\lambda^a = \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j}}$ and $\lambda^b = \frac{\bar{U}_j}{\frac{1-\gamma}{2} f_{i,j}}$.

Plugging the optimal solution back to the incentive constraints of participants and objective function, we reach the results in lemma. ■

It turns on a competitive search equilibrium must be separating, if the joint value $f_{i,j} \neq f_{i,j'}$ for a fixed i and $j \neq j'$. This is the assumption made in models from literature such as Eeckhout and Kircher (2010). This result comes from that a non-separating market can be viewed as a collection of separating markets, with an additional restriction that all of these separating markets must share the identical contact rates. When $f_{i,j}$ varies in the matching partners' types (dimension j), this identical contact rate constraint is binding and strictly decreases payoffs for the platform.

Lemma 13 *If $f_{i,j} \neq f_{i,j'}$ for $j \neq j'$, any competitive search equilibrium with observable types is separating.*

Proof. We first replace (ϕ_i^a, ϕ_j^b) in the objective function by the constraints in problem (37): The original problem is now an unconstrained problem in terms of (ϕ^a, ϕ^b) and (ω_i^a, ω_j^b):

$$\max_{\boldsymbol{\lambda}>0, \boldsymbol{G}^s \in \Delta^I} m(\lambda^a, \lambda^b) \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right).$$

Given any $\bar{\boldsymbol{U}}$, we have can write the value of objective function as:

$$\sum_{i=1}^{I} \sum_{j=1}^{I} \omega_i^a \omega_j^b m(\lambda^a, \lambda^b) \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right) \leq \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b V_{i,j}^*,$$

where the inequality comes from the fact $V_{i,j}^*$ is a solution to problem (38) and any positive (λ^a, λ^b) is feasible for (38). The inequality holds with equality only if $\lambda^s = \lambda_{i,j}^s$ for any (i,j) such that $G_i^s G_j^{-s} > 0$.

Suppose there is a CSE that is non-separating while $f_{i,j}$ is strictly increasing in j. WLOG, assume there is $\tau \in T$ such that $\omega_j^b > 0$, $\omega_{j'}^b > 0$, and $j \neq j'$. From lemma 12, for some i such that $G_i^a > 0$:

$$\lambda_{i,j}^a = \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j}} \neq \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j'}} = \lambda_{i,j'}^a.$$

This means λ^a cannot equal to both contact rates in separating markets. This means:

$$c = \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right) < \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b V_{i,j}^*.$$

Thus one of the separating markets such that $\omega_i^a \omega_j^b > 0$ must yield strictly higher payoff for the platform. This contradicts to the second condition of the definition for a partial equilibrium (,and thus competitive search equilibrium). ■

Lemma 13 implies that we should only look for separating competitive search equilibrium. From here on we use the enumeration defined in section 2.5, and index markets by the pair that shows up on the two sides. Given the results in Lemma 12, the step-two problem is to find the set of (i,j) that maximize platforms' value:

$$\bar{V} = \max_{i,j} V_{i,j}. \tag{39}$$

Naturally, the partial-equilibrium set of markets M is the set of maximizers to problem (39).

In the next proposition, we consider the CES matching function and show our results resembles the ones in Eeckhout and Kircher (2010), although in our context the matching function takes three parties.

C.4 Proof for Proposition 4

From (39), $M = \arg\max_{i,j} V_{i,j}$. This is a maximization problem on a finite set. Thus a solution (and correspondingly a partial equilibrium) must exist given $\bar{\boldsymbol{U}}$. We now show (1) M must have stated sorting pattern and (2) we can construct a unique competitive search equilibrium. From lemma 12:

$$V_{i,j} = \partial_P M(n^a, n^b, 1)(u_{i,j} + u_{j,i}),$$

where (n^a, n^b) solves

$$\partial_{n^a} M(n^a, n^b, 1)(u_{i,j} + u_{j,i}) = \bar{U}_i,\ \partial_{n^b} M(n^a, n^b, 1)(u_{i,j} + u_{j,i}) = \bar{U}_j.$$

Using the function form we find:

$$\left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-\frac{1}{\gamma}-1} (n^a)^{-\gamma-1} = \frac{\bar{U}_i}{u_{i,j} + u_{j,i}}$$

$$\left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-\frac{1}{\gamma}-1} (n^a)^{-\gamma-1} = \frac{\bar{U}_j}{u_{i,j} + u_{j,i}}$$

Raising both side to the power of $\frac{\gamma}{1+\gamma}$:

$$\left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-1} (n^a)^{-\gamma} = \left(\frac{\bar{U}_i}{u_{i,j} + u_{j,i}}\right)^{\frac{\gamma}{1+\gamma}}$$

$$\left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-1} (n^b)^{-\gamma} = \left(\frac{\bar{U}_j}{u_{i,j} + u_{j,i}}\right)^{\frac{\gamma}{1+\gamma}}$$

Summing them:

$$1 - \left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-1} = \left(\frac{\bar{U}_i}{u_{i,j} + u_{j,i}}\right)^{\frac{\gamma}{1+\gamma}} + \left(\frac{\bar{U}_j}{u_{i,j} + u_{j,i}}\right)^{\frac{\gamma}{1+\gamma}}$$

Thus:

$$\left((n^a)^{-\gamma} + (n^b)^{-\gamma} + 1\right)^{-1} = 1 - \left(\frac{\bar{U}_i}{u_{i,j} + u_{j,i}}\right)^{\frac{\gamma}{1+\gamma}} - \left(\frac{\bar{U}_j}{u_{i,j} + u_{j,i}}\right)^{\frac{\gamma}{1+\gamma}}$$

and

$$\partial_P M(n^a, n^b, 1) = \left[1 - \left(\frac{\bar{U}_i}{u_{i,j}+u_{j,i}}\right)^{\frac{\gamma}{1+\gamma}} - \left(\frac{\bar{U}_j}{u_{i,j}+u_{j,i}}\right)^{\frac{\gamma}{1+\gamma}}\right]^{\frac{1+\gamma}{\gamma}}$$

$$V_{i,j} = \left[f_{i,j}^{\frac{\gamma}{1+\gamma}} - \bar{U}_i^{\frac{\gamma}{1+\gamma}} - \bar{U}_j^{\frac{\gamma}{1+\gamma}}\right]^{\frac{1+\gamma}{\gamma}}$$

If $f_{i,j}^{\frac{\gamma}{1+\gamma}}$ is supermodular, then for any (i,j) such that $i \neq j$:

$$V_{i,i}^{\frac{\gamma}{1+\gamma}} + V_{j,j}^{\frac{\gamma}{1+\gamma}} > 2V_{i,j}^{\frac{\gamma}{1+\gamma}},$$

for any $\bar{U}_i$ and $\bar{U}_j$.

If $f_{i,j}$ is log-supermodular. We prove PAM by contradiction. More precisely, we want to show that if $(i,j) \in M$, then $i = j$. Suppose otherwise. The formula of $V_{i,j}$ implies

$$\frac{V_{i,j}V_{j,j}}{V_{i,j}^2} = \left(\frac{f_{i,i}f_{j,j}}{f_{i,j}^2}\right)^{\frac{1}{\gamma}} > 1.$$

Thus $\max\{V_{i,i}, V_{j,j}\} > V_{i,j}$. It is profitable for the platforms to deviate from the focal active market to either a market with (i,i) or a market (j,j). A contradiction to the optimality. So any separating partial equilibrium has markets with identical types on both sides, which satisfies the definition of positive assortative matching.

We then show there is a unique separating competitive search equilibrium. In a competitive search equilibrium, $\bar{V} = V_{i,i} = c$:

$$c = V_{i,j} = \gamma\left(\frac{1-\gamma}{2}\right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i\bar{U}_j\right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}}.$$

Solving this equation gives a unique value of $\bar{U}_i$. With this value of $\bar{U}_i$, there is a unique number of participants per posting $n_{i,i}^s$ from Lemma 12. Lastly, from the market to clear for type i, we compute $N^P(m_{i,i}) = \frac{\bar{\omega}_i}{2n_{i,i}}$. This construction leads to unique separating competitive search equilibrium with positive assortative matching.

If $f_{i,j}$ is log-submodular. Suppose there are two active markets for (i_1, j_1) and (i_2, j_2) and $i_1 > i_2$, $j_1 > j_2$.

$$\frac{V_{i_1,j_2}V_{i_2,j_1}}{V_{i_1,j_1}V_{i_2,j_2}} = \left(\frac{f_{i_1,j_2}f_{i_2,j_1}}{f_{i_1,j_1}f_{i_2,j_2}}\right)^{\frac{1}{\gamma}} > 1.$$

This contradicts to optimality of (i_1, j_1) or (i_2, j_2).

The negative assortative matching in M and the clear marketing implies that all active markets have the pairs in the form of $(1, I)$, $(2, I-1)$, ..., $(I/2, I/2+1)$ if I is even, and in the form of $(1, I)$, $(2, I-1)$, ..., $((I-1)/2, (I-1)/2)$ if I is odd. Take each of these pair (i, j), we clear the market in the following order. First solve $\bar{U}_i$ and $\bar{U}_j$ as solution to:

$$c = \gamma \left(\frac{1-\gamma}{2} \right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i \bar{U}_j \right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}},$$

$$\frac{\bar{\omega}_i}{\bar{\omega}_j} = \frac{\bar{U}_i}{\bar{U}_j}.$$

This is a system of two equations with unique solution $(\bar{U}_i, \bar{U}_j)$. We can solve from lemma 12 the correspondingly $n_{i,j}^s$. The measure of postings is given by $N_{i,j}^P = N^P(i,j) = \frac{\bar{\omega}_i}{n_{i,j}^a}$.