Online Appendix for "Competitive Sorting with Bilateral Private Information"

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F Existence of Competitive Search Equilibrium

F.1 Statement and Roadmap

We first define strict IWTP:

Assumption 9 (Strict IWTP) *$\forall\, s \in \{a, b\}$ $i > i' \in \mathbb{I}^s$, $j \in \mathbb{I}^{\bar{s}}$, $u^s(i, j) > u^s(i', j)$.*

Combining this with Common Ranking, Supermodularity, a finite type space, and a positive intermediation cost delivers a general existence theorem:[21]

Theorem 1 *Assume Common Ranking, Supermodularity, and strict IWTP. If $c > 0$ and $\mathbb{I}^a, \mathbb{I}^b$ are finite, a CSE exists.*

The proof uses a Kakutani fixed-point argument on the pair (U, μ), where μ is the match measure of Definition 2. At given equilibrium utility U, platforms choose μ to concentrate match mass on profit-maximizing markets, and a Walrasian auctioneer adjusts U in the direction of excess demand, in the spirit of Gale (1955), Debreu (1956), and Nikaido (1956). The technical crux is that the platform's value $\hat{V}(k^a, k^b; U)$ from attracting (k^a, k^b) is continuous in U up to the boundary where the market becomes infeasible, where $\hat{V} \to 0$.

Strict IWTP is what delivers boundary continuity. Because $u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}}) > 0$ for every $i < k^s$, downward incentive constraints act uniformly as a strictly positive lower bound on λ^s. Any way the feasible set becomes empty therefore drives either $m \to 0$ or $\phi^a + \phi^b \to 0$, with $\hat{V} \to 0$ in either case. The Kakutani argument then runs cleanly.

Outside IWTP, this boundary continuity fails, and existence can fail with it. The failures take qualitatively different forms. Under strict DWTP, the coefficient $u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}})$ is uniformly negative, so downward incentive constraints act as an upper bound on λ^s.

[21] The finite-type assumption delivers compactness. Extending the proof to a continuous type space would require additional functional-analytic machinery beyond the scope of this paper. Sections 5–7 of the main text construct continuous-type equilibria directly and do not rely on Theorem 1.

At $U^s(k^s) = 0$, this upper bound collapses to zero, but the platform retains a profitable degenerate deviation: a market with $\lambda^s = 0$, fees set to the rationality bound, and matching probability $m(0,0)$, earning value $m(0,0)f(k^a,k^b) > 0$. Any perturbation of $U^s(k^s)$ off zero destroys this deviation, and the resulting discontinuity in $\hat{V}$ at the corner of $\mathcal{U}$ defeats the fixed-point argument. Section F.5 exhibits a two-type symmetric example in which no CSE exists for an open interval of intermediation costs.[22] The corner-driven character of this failure suggests that additional structure ruling out boundary equilibria, such as a participation condition guaranteeing positive utility for all types, may rescue existence under DWTP. We leave a general result in this direction to future work.

Under non-monotone WTP, the failure is interior rather than boundary. With $u^s(k^s,k^{\bar{s}}) - u^s(i,k^{\bar{s}})$ taking different signs across types i, the downward incentive constraints from different types impose conflicting bounds on λ^s, lower bounds from some types and upper bounds from others, that can cross in the interior of $\mathbb{\Lambda}$. The discontinuity in $\hat{V}$ then occurs at strictly positive utility values rather than at the corner of $\mathcal{U}$, and the mechanism is robust to perturbations of the example. Section F.6 exhibits a three-type symmetric example. We see no analogous structural fix.

Strictness of IWTP, positivity of c, and finiteness of the type space play distinct roles in the proof, which we summarize before turning to the construction. The proof establishes existence in four steps: (i) reduce the analysis of active markets to a finite-dimensional program indexed by the target types (k^a,k^b), with value $\hat{V}(k^a,k^b;U)$; (ii) define a platform correspondence (which chooses μ given U) and a Walrasian-auctioneer correspondence (which adjusts U in the direction of excess demand); (iii) apply Kakutani's theorem to obtain a fixed point (U^*,μ^*); (iv) verify that (U^*,μ^*) satisfies Definition 2. The three assumptions of the theorem play distinct roles. Strict IWTP makes the downward incentive constraints a *uniform* positive lower bound on λ^s, which is what delivers continuity of $\hat{V}$ up to the boundary of $\mathcal{U}$ (Lemma 11). A positive c ensures that $A(U)$ (the set of free-entry markets at U) is closed and that $m > 0$ at every supported market when $V^*(U^*) = c$. Finiteness of $\mathbb{I}^a, \mathbb{I}^b$ makes the domain $\mathcal{U}$ a finite-dimensional polytope and the measure space $\mathcal{M}$ compact.

F.2 Setup

Fix a constant $C > \max_{i\in\mathbb{I}^a,\, j\in\mathbb{I}^b}\left[u^a(i,j) + u^b(j,i)\right]$ and set

$$\mathcal{U} \;\equiv\; [0,C]^{|\mathbb{I}^a|} \times [0,C]^{|\mathbb{I}^b|},$$

[22] The applications in Sections 6 and 7.2 of the main text construct DWTP equilibria directly and do not rely on Theorem 1.

a compact convex polytope. We will prove existence of a CSE with equilibrium utility $U \in \mathcal{U}$.

Next, fix $(k^a, k^b) \in \mathbb{I}^a \times \mathbb{I}^b$ and $U \in \mathcal{U}$. By Proposition 1, a separating terms-of-trade attracting (k^a, k^b) solves

$$\hat{V}(k^a, k^b; U) \;=\; \sup_{\lambda^a, \lambda^b, \phi^a, \phi^b} \; m(\lambda^a, \lambda^b)(\phi^a + \phi^b) \qquad \text{subject to } (\lambda^a, \lambda^b, \phi^a, \phi^b) \in L(k^a, k^b; U), \tag{29}$$

where $L(k^a, k^b; U) \subset \mathbb{R}^4$ denotes the feasible set, the combinations of $(\lambda^a, \lambda^b, \phi^a, \phi^b)$ satisfying

$$\lambda^s\big(u^s(k^s, k^{\bar{s}}) - \phi^s\big) = U^s(k^s), \quad s = a, b; \tag{P}$$

$$U^s(i) \geq \lambda^s\big(u^s(i, k^{\bar{s}}) - \phi^s\big), \quad i < k^s, \; s = a, b; \tag{IC}$$

$$\phi^a + \phi^b \geq 0; \tag{F}$$

$$\phi^s \leq u^s(k^s, k^{\bar{s}}), \quad s = a, b; \tag{B}$$

$$(\lambda^a, \lambda^b) \in \mathbb{\Lambda}.$$

The bounds (F) and (B) confine ϕ^s to the compact interval $[-u^{\bar{s}}(k^{\bar{s}}, k^s),\, u^s(k^s, k^{\bar{s}})]$ and $\mathbb{\Lambda}$ is compact, so L sits inside a compact set. When $L(k^a, k^b; U) = \emptyset$, no feasible terms-of-trade attracts (k^a, k^b), and we adopt the convention $\hat{V}(k^a, k^b; U) = 0$: this is consistent with the platform's outside option of not operating this market (which earns gross profit zero), and makes $\hat{V}(\cdot; U)$ upper semicontinuous up to the boundary of the feasibility region (Lemma 11).

For $\lambda^s > 0$, use (P) to eliminate ϕ^s from the remaining constraints:

$$U^s(k^s) - U^s(i) \;\leq\; \lambda^s\big[u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}})\big], \quad i < k^s, \; s = a, b; \tag{IC$'$}$$

$$\frac{U^a(k^a)}{\lambda^a} + \frac{U^b(k^b)}{\lambda^b} \;\leq\; f(k^a, k^b) \;\equiv\; u^a(k^a, k^b) + u^b(k^b, k^a); \tag{F$'$}$$

and (B) reduces to $U^s(k^s)/\lambda^s \geq 0$, automatic. At $\lambda^s = 0$, (P) forces $U^s(k^s) = 0$ and ϕ^s is free within $[-u^{\bar{s}}(k^{\bar{s}}, k^s),\, u^s(k^s, k^{\bar{s}})]$.

Under strict IWTP, the coefficient $u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}})$ on the right side of (IC$'$) is strictly positive for every $i < k^s$, so (IC$'$) acts uniformly on each side as a strictly positive lower bound on λ^s. This uniform-sign property, which would fail under weak IWTP or under non-Monotone WTP, is what drives the continuity arguments below.

F.3 Continuity of $\hat{V}$

This subsection establishes the technical core of the existence proof: the platform's value function V^* is continuous on $\mathcal{U}$, and the argmax in (29) has closed graph. Throughout, we maintain Common Ranking, Supermodularity, and strict IWTP.

Lemma 11 *Assume Common Ranking, Supermodularity, and strict IWTP. Then* $V^*(U) \equiv \max_{(k^a,k^b)\in\mathbb{I}^a\times\mathbb{I}^b} \hat{V}(k^a,k^b;U)$ *is continuous on* $\mathcal{U}$*, each* $\hat{V}(k^a,k^b;\cdot)$ *is upper semicontinuous on* $\mathcal{U}$*, and for every* $U \in \mathcal{U}$ *with* $L(k^a,k^b;U) \neq \emptyset$*, the set of maximizers in Problem (29) is nonempty and has closed graph in* U*.*

The proof proceeds through two intermediate lemmas: Lemma 12 establishes that the feasibility correspondence has closed graph, and Lemma 13 establishes that locally feasible points at one U can be perturbed to stay feasible at nearby U'. Together they deliver continuity of V^*.

Lemma 12 *The graph* $\{(U,\lambda,\phi) \in \mathcal{U}\times\mathbb{R}^4 : (\lambda,\phi) \in L(k^a,k^b;U)\}$ *is closed. For every* $U \in \mathcal{U}$ *with* $L(k^a,k^b;U) \neq \emptyset$*, the set of solutions to problem (29) is nonempty, compact, and has closed graph in* U*. Moreover,* $\hat{V}(k^a,k^b;\cdot)$ *is upper semicontinuous on* $\mathcal{U}$*.*

Proof of Lemma 12. The constraint $(\lambda^a,\lambda^b) \in \mathbb{\Lambda}$ cuts out a closed subset of $\mathcal{U}\times\mathbb{R}^4$ because $\mathbb{\Lambda}$ is closed. Each of (P), (IC), (F), (B) is a continuous (in)equality in (λ,ϕ,U), hence cuts out a closed subset. The graph is the intersection of these closed sets, hence closed.

The compact envelope $\mathbb{\Lambda}\times\prod_s[-u^{\bar{s}}(k^{\bar{s}},k^s), u^s(k^s,k^{\bar{s}})]$ makes $L(k^a,k^b;U)$ compact for each U, and the objective $m(\lambda)(\phi^a+\phi^b)$ is continuous, so the argmax is nonempty (Weierstrass) and closed. Closed-graph property of the argmax follows from closed graph of L plus continuity of the objective by Berge's theorem.

For upper semicontinuity of $\hat{V}(k^a,k^b;\cdot)$: let $U_n \to U$. If infinitely many U_n have $L(k^a,k^b;U_n) = \emptyset$, then $\hat{V}(k^a,k^b;U_n) = 0 \leq \hat{V}(k^a,k^b;U)$ along that subsequence. For the remaining U_n, pick maximizers $(\lambda_n,\phi_n) \in L(U_n)$; a convergent subsequence has limit $(\lambda_*,\phi_*) \in L(U)$ by closed graph, hence $\hat{V}(k^a,k^b;U_n) \to m(\lambda_*)(\phi_*^a+\phi_*^b) \leq \hat{V}(k^a,k^b;U)$. So $\limsup \hat{V}(k^a,k^b;U_n) \leq \hat{V}(k^a,k^b;U)$. ■

Lemma 13 *Let* $U \in \mathcal{U}$*,* $(k^a,k^b) \in \mathbb{I}^a\times\mathbb{I}^b$*, and* $(\lambda^*,\phi^*) \in L(k^a,k^b;U)$ *with* $m(\lambda^*) > 0$ *and* $\phi^{*a}+\phi^{*b} > 0$*. For* $\epsilon > 0$*, set* $\lambda'^s = \max\{\lambda^{*s},\epsilon\}$ *on each side, and* $\phi'(U'')^s = u^s(k^s,k^{\bar{s}}) - U''^s(k^s)/\lambda'^s$ *for* $U'' \in \mathcal{U}$*. Under strict IWTP, for* $\epsilon > 0$ *sufficiently small:*

1. $(\lambda',\phi'(U)) \in L(k^a,k^b;U)$*;*
2. $m(\lambda')(\phi'(U)^a+\phi'(U)^b) \to m(\lambda^*)(\phi^{*a}+\phi^{*b})$ *as* $\epsilon \to 0$*;*
3. *there exists a neighborhood* N *of* U *in* $\mathcal{U}$ *such that* $(\lambda',\phi'(U'')) \in L(k^a,k^b;U'')$ *for every* $U'' \in N$*.*

Proof of Lemma 13. $\lambda^* \in \mathbb{A}$ with $m(\lambda^*) > 0$ and m continuous strictly decreasing imply λ^* lies in the set $\{m > 0\} \cap \mathbb{A}$. For ϵ small, λ' differs from λ^* only on sides with $\lambda^{*s} = 0$, and only by a small amount ϵ, so $\lambda' \in \mathbb{A}$ with $m(\lambda') > 0$. We verify each constraint at $(\lambda', \phi'(U), U)$.

(P) *at* (k^a, k^b)*.* Holds by construction of $\phi'(U)$: $\lambda'^s(u^s(k^s, k^{\bar{s}}) - \phi'(U)^s) = \lambda'^s \cdot U^s(k^s)/\lambda'^s = U^s(k^s)$.

(F) *and* (B). On sides with $\lambda^{*s} > 0$: $\lambda'^s = \lambda^{*s}$ and $\phi'(U)^s = \phi^{*s}$ (identical to the original maximizer), so these constraints hold as at (λ^*, ϕ^*). On sides with $\lambda^{*s} = 0$: (P) at (λ^*, ϕ^*) forces $U^s(k^s) = 0$, so $\phi'(U)^s = u^s(k^s, k^{\bar{s}})$, saturating (B) with equality. (F) then reads $u^s(k^s, k^{\bar{s}}) + \phi'(U)^{\bar{s}} \geq 0$; on side $\bar{s}$, $\phi'(U)^{\bar{s}}$ equals $\phi^{*\bar{s}}$ if $\lambda^{*\bar{s}} > 0$, or $u^{\bar{s}}(k^{\bar{s}}, k^s)$ if $\lambda^{*\bar{s}} = 0$. In the first subcase, $\phi^{*a} + \phi^{*b} > 0$ gives $\phi^{*\bar{s}} > -u^s(k^s, k^{\bar{s}})$ (from (F) at (λ^*, ϕ^*) when ϕ^{*s} equals its upper bound $u^s(k^s, k^{\bar{s}})$), so (F) holds. In the second subcase, $\phi'(U)^s + \phi'(U)^{\bar{s}} = u^s(k^s, k^{\bar{s}}) + u^{\bar{s}}(k^{\bar{s}}, k^s) = f(k^s, k^{\bar{s}}) > 0$.

(IC) *at* (k^a, k^b)*, side* s*, type* $i < k^s$*.* Under strict IWTP, $c_i \equiv u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}}) > 0$. Via (IC′), the IC is $U^s(k^s) - U^s(i) \leq \lambda'^s \cdot c_i$. When $\lambda^{*s} > 0$: $\lambda'^s = \lambda^{*s}$, and the IC at (λ^*, ϕ^*) gives the result directly. When $\lambda^{*s} = 0$: $U^s(k^s) = 0$, so the IC reads $-U^s(i) \leq \epsilon c_i$, i.e., $U^s(i) \geq -\epsilon c_i$. This holds because $U^s(i) \geq 0$ (box constraint on $\mathcal{U}$).

Extension to neighborhood. The above establishes (1); for (3), fix ϵ small and consider $U'' \in \mathcal{U}$ close to U. (P) at $(k^a, k^b, U'', \lambda', \phi'(U''))$ holds by construction. For (F) and (B): on sides with $\lambda^{*s} > 0$, $\phi'(U'')^s \to \phi^{*s}$ continuously as $U'' \to U$, so the constraints hold for U'' close enough. On sides with $\lambda^{*s} = 0$: $\phi'(U'')^s = u^s(k^s, k^{\bar{s}}) - U''^s(k^s)/\epsilon$. As $U'' \to U$ (with $U^s(k^s) = 0$), $U''^s(k^s) \to 0$, so $\phi'(U'')^s \to u^s(k^s, k^{\bar{s}})$; (B) holds with small slack, and (F) reads $\phi'(U'')^s + \phi'(U'')^{\bar{s}} \to f(k^s, k^{\bar{s}}) > 0$. For (IC): the IC at U'' reads $U''^s(k^s) - U''^s(i) \leq \lambda'^s \cdot c_i$. When $\lambda^{*s} > 0$: $\lambda'^s = \lambda^{*s}$, the IC at (λ^*, ϕ^*, U) gives $U^s(k^s) - U^s(i) \leq \lambda^{*s} c_i$, and continuity of LHS in U'' preserves the inequality with small slack for U'' close to U (the RHS doesn't change). When $\lambda^{*s} = 0$: $\lambda'^s = \epsilon$, and the IC reads $U''^s(k^s) - U''^s(i) \leq \epsilon c_i$. As $U'' \to U$, LHS $\to U^s(k^s) - U^s(i) = 0 - U^s(i) \leq 0$ (since $U^s(i) \geq 0$), so for U'' close enough, LHS $\leq \epsilon c_i/2 < \epsilon c_i$.

Convergence of values. On sides with $\lambda^{*s} > 0$: $\lambda'^s = \lambda^{*s}$ and $\phi'(U)^s = \phi^{*s}$, so those terms are unchanged. On sides with $\lambda^{*s} = 0$: $\phi'(U)^s = \phi^{*s} = u^s(k^s, k^{\bar{s}})$ (the extremal fee; ϕ^{*s} is free in its range at $\lambda^{*s} = 0$, and we may assume the selected maximizer places ϕ^{*s} there as it maximizes $\phi^a + \phi^b$). As $\epsilon \to 0$, $m(\lambda') \to m(\lambda^*)$ by continuity. The value $m(\lambda')(\phi'(U)^a + \phi'(U)^b) \to m(\lambda^*)(\phi^{*a} + \phi^{*b})$. ■

Proof of Lemma 11. Upper semicontinuity of each $\hat{V}(k^a, k^b; \cdot)$ on $\mathcal{U}$ is Lemma 12, as is nonemptiness and closed graph of the argmax. Upper semicontinuity of the finite max V^*

follows from USC of each $\hat{V}(k^a, k^b; \cdot)$.

For lower semicontinuity of V^* on $\mathcal{U}$: let $U_n \to U$ in $\mathcal{U}$. If $V^*(U) = 0$, trivial. Otherwise pick $(k^a, k^b) \in \arg\max \hat{V}(\cdot; U)$ with $\hat{V}(k^a, k^b; U) = V^*(U) > 0$ and a maximizer $(\lambda^*, \phi^*) \in \arg\max$ (29); since $\hat{V}(k^a, k^b; U) > 0$ we have $m(\lambda^*) > 0$ and $\phi^{*a} + \phi^{*b} > 0$. Given $\delta > 0$, Lemma 13 supplies a perturbed feasible point $(\lambda', \phi'(\cdot))$ with value at least $V^*(U) - \delta$ (for ϵ small), along with a neighborhood N of U in $\mathcal{U}$ on which $(\lambda', \phi'(U'')) \in L(k^a, k^b; U'')$. For $U_n \in N$:

$$V^*(U_n) \;\geq\; \hat{V}(k^a, k^b; U_n) \;\geq\; m(\lambda')\big(\phi'(U_n)^a + \phi'(U_n)^b\big) \;\xrightarrow{n\to\infty}\; m(\lambda')\big(\phi'(U)^a + \phi'(U)^b\big) \;\geq\; V^*(U) - \delta.$$

Since $\delta > 0$ was arbitrary, $\liminf V^*(U_n) \geq V^*(U)$. Continuity of V^* on $\mathcal{U}$ follows. ■

F.4 Proof of Theorem 1

Proof of Theorem 1. Define the argmax set of markets achieving free entry,

$$A(U) \;\equiv\; \{(k^a, k^b) : L(k^a, k^b; U) \neq \emptyset \text{ and } \hat{V}(k^a, k^b; U) = c\}.$$

By Lemma 11, V^* is continuous and each $\hat{V}(k^a, k^b; \cdot)$ is upper semicontinuous on $\mathcal{U}$. Combining the two, $U \mapsto \arg\max_{(k^a,k^b)} \hat{V}(\cdot; U)$ is upper hemicontinuous on $\mathcal{U}$: if $(k^a, k^b) \in \arg\max(U_n)$ with $U_n \to U$, then $\hat{V}(k^a, k^b; U_n) = V^*(U_n)$; taking lim sup, upper semicontinuity gives $\hat{V}(k^a, k^b; U) \geq V^*(U)$, and $\hat{V} \leq V^*$ gives equality. The same argument with c in place of V^* gives upper hemicontinuity of A on $\mathcal{U}$.

Platform and auctioneer correspondences Throughout, μ denotes the match measure of Definition 2. Side-s agent flows are recovered from μ by the Consistency condition: at a separating market (k^a, k^b) with matching probability $\lambda^{s*} > 0$, agent flow on side s is $\mu(k^a, k^b)/\lambda^{s*}$. Match measure is uniformly bounded by total population, which makes the fixed-point argument tractable.

Pick $\bar{\mu} > \sum_{s,i} I^s F^s(\{i\})$ strictly exceeding total population on both sides combined—any market-clearing allocation has $\sum \mu \leq \min_s I^s$, so this upper bound is never binding at a CSE—and set $\mathcal{M} \equiv [0, \bar{\mu}]^{|\square^a| \times |\square^b|}$.

For each (k^a, k^b) with $L(k^a, k^b; U) \neq \emptyset$, Lemma 11 gives that $\arg\max$ (29) is nonempty, compact, and has closed graph in U. The Kuratowski–Ryll-Nardzewski measurable selection theorem (Kuratowski and Ryll-Nardzewski, 1965) then supplies a measurable selection $(\lambda^*(k^a, k^b; \cdot), \phi^*(k^a, k^b; \cdot))$. We pick this selection to satisfy: $\lambda^{s*}(k^a, k^b; U) > 0$ *whenever* $U^s(k^s) > 0$. This property is automatic at every point of the argmax, because (P) requires

$\lambda^s(u^s - \phi^s) = U^s(k^s)$, which forces $\lambda^s > 0$ when $U^s(k^s) > 0$.[23] Combining (P) and (B), whenever $U^s(k^s) > 0$,

$$\lambda^{s*}(k^a, k^b; U) \;=\; \frac{U^s(k^s)}{u^s(k^s, k^{\bar{s}}) - \phi^{s*}} \;\geq\; \frac{U^s(k^s)}{u^s(k^s, k^{\bar{s}}) + u^{\bar{s}}(k^{\bar{s}}, k^s)} \;=\; \frac{U^s(k^s)}{f(k^s, k^{\bar{s}})},$$

so λ^{s*} is bounded away from zero on any subset of $\mathcal{U}$ where $U^s(k^s)$ is bounded away from zero.

The *platform correspondence* $\Gamma_\mu : \mathcal{U} \rightrightarrows \mathcal{M}$ is

$$\Gamma_\mu(U) = \begin{cases} \left\{\mu \in \mathcal{M} : \operatorname{supp}(\mu) \subseteq \arg\max_{(k^a,k^b)} \hat{V}(\cdot; U),\ \sum \mu = \bar{\mu}\right\} & V^*(U) > c, \\ \left\{\mu \in \mathcal{M} : \operatorname{supp}(\mu) \subseteq A(U)\right\} & V^*(U) = c, \\ \{0\} & V^*(U) < c. \end{cases}$$

Γ_μ is nonempty, compact- and convex-valued, and upper hemicontinuous: the three cases are locally mutually exclusive in U by continuity of V^*, and the support-containment constraint is closed by upper hemicontinuity of the argmax correspondence and of A.

For each (s, i), define *demand* and *excess demand*:

$$D^s(i; U, \mu) \equiv \sum_{\substack{(k^a,k^b):\, k^s = i \\ \lambda^{s*}(k^a,k^b;U) > 0}} \frac{\mu(k^a, k^b)}{\lambda^{s*}(k^a, k^b; U)}, \qquad Z^s(i; U, \mu) \equiv D^s(i; U, \mu) - I^s F^s(\{i\}).$$

This agrees with the side-s agent flow ν^s of Definition 2: at a separating market (k^a, k^b) with $k^s = i$, the Consistency condition $d\mu = \lambda^{s*}\, d\nu^s$ gives type-i flow $\mu(k^a, k^b)/\lambda^{s*}$, which is the summand in D^s. By the bound on λ^{s*} above, $D^s(i; U, \mu)$ is continuous in (U, μ) on $\{U : U^s(i) > 0\}$.

The *auctioneer correspondence* $\Gamma_U : \mathcal{U} \times \mathcal{M} \rightrightarrows \mathcal{U}$ is

$$\Gamma_U(U, \mu) \;=\; \arg\max_{U' \in \mathcal{U}} \sum_{s,i} U'^s(i)\, Z^s(i; U, \mu).$$

The auctioneer raises $U'^s(i)$ toward C when type i has excess demand and lowers it toward 0 when type i has excess supply, in the spirit of the Gale–Debreu–Nikaido lemma.

We now apply a Kakutani fixed-point argument. The joint correspondence $\Gamma(U, \mu) \equiv \Gamma_U(U, \mu) \times \Gamma_\mu(U)$ has nonempty, convex, compact values (linearity of the auctioneer's ob-

[23] When $U^s(k^s) = 0$, the argmax may admit both $\lambda^s = 0$ and, if the matching function satisfies $m(0^+, \lambda^{\bar{s}}) > m(\lambda^s, \lambda^{\bar{s}})$ strictly, only $\lambda^s = 0$ is optimal. The selection takes $\lambda^{s*} = 0$ there; nothing in the argument below depends on the tie-breaking choice.

jective and compactness of $\mathcal{U}$; convexity of the support-containment constraint in μ and compactness of $\mathcal{M}$). The remaining hypothesis is closed graph, which holds on the interior domain $\mathcal{U}^\circ \equiv \{U \in \mathcal{U} : U^s(i) > 0 \text{ for all } (s,i)\}$: on $\mathcal{U}^\circ \times \mathcal{M}$, the bound $\lambda^{s*} \geq U^s(k^s)/f$ gives a uniform positive lower bound, D^s is continuous, Z^s is continuous, and the linear auctioneer's objective inherits continuity, so Berge's theorem delivers closed graph of the argmax.

At boundary points where $U^s(i) = 0$ for some (s,i), the demand $D^s(i;\cdot,\cdot)$ can be discontinuous: along sequences $U^s_n(k^s) \downarrow 0$ with μ_n bounded away from zero at some market with $k^s = i$, the selected $\lambda^{s*}_n \downarrow 0$ drives μ_n/λ^{s*}_n unbounded, whereas the limit selection at $U^s(k^s) = 0$ has $\lambda^{s*} = 0$ and contributes nothing to $D^s(i;U,\mu)$. To handle the boundary, we use a standard perturbation: for $\eta \in (0, \min_{s,i} I^s F^s(\{i\})/2)$, let $\mathcal{U}_\eta \equiv [\eta, C]^{|\mathbb{I}^a|+|\mathbb{I}^b|}$. On $\mathcal{U}_\eta$, the lower bound $\lambda^{s*} \geq \eta/f$ is uniform, so D^s is continuous, Γ_U has closed graph by Berge, and Kakutani's theorem applied to Γ restricted to $\mathcal{U}_\eta \times \mathcal{M}$ delivers a fixed point (U^*_η, μ^*_η).[24]

Take a sequence $\eta_n \downarrow 0$. By compactness of $\mathcal{U} \times \mathcal{M}$, the sequence $(U^*_{\eta_n}, \mu^*_{\eta_n})$ has a convergent subsequence; call the limit $(U^*, \mu^*) \in \mathcal{U} \times \mathcal{M}$. We will verify that (U^*, μ^*) satisfies the conditions needed to build a CSE.

Verification We show that the limit point (U^*, μ^*) satisfies conditions (i)–(iii) of Proposition 1 Part 2. Throughout, $(U^*_{\eta_n}, \mu^*_{\eta_n}) \to (U^*, \mu^*)$ along the chosen subsequence, and we abbreviate $\eta \equiv \eta_n$ when context is clear.

Step 1: $V^*(U^*) \leq c$. Suppose for contradiction $V^*(U^*) > c$. By continuity of V^* (Lemma 11), $V^*(U^*_\eta) > c$ for all sufficiently small η, so the platform correspondence puts $\sum \mu^*_\eta = \bar{\mu}$ on the tail. At any argmax market for such η, $V^*(U^*_\eta) > 0$ forces $m(\lambda^*) > 0$ and $\phi^{a*} + \phi^{b*} > 0$; combining with (P), at least one side s has $U^{*s}_\eta(k^s) > 0$, so the selection rule gives $\lambda^{s*} > 0$ on that side. Counting contributions to demand:

$$\sum_s \sum_i D^s(i; U^*_\eta, \mu^*_\eta) \;=\; \sum_s \sum_{(k^a,k^b):\lambda^{s*}>0} \frac{\mu^*_\eta(k^a,k^b)}{\lambda^{s*}} \;\geq\; \sum_{(k^a,k^b)} \mu^*_\eta(k^a,k^b) \;=\; \bar{\mu}, \tag{30}$$

where the inequality uses $\lambda^{s*} \leq 1$ on the contributing side and the fact that at least one side contributes at each market in the support. Since $\bar{\mu} > \sum_{s,i} I^s F^s(\{i\})$, this gives $\sum_{s,i} Z^s(i; U^*_\eta, \mu^*_\eta) > 0$, so $Z^s(i; U^*_\eta, \mu^*_\eta) > 0$ for some (s,i). The auctioneer on $\mathcal{U}_\eta$ with a positive coefficient on $U'^s(i)$ then sets $U^{*s}_\eta(i) = C$.

But $U^{*s}_\eta(i) = C$ is incompatible with feasibility at any market serving type i: from (P)

[24] On $\mathcal{U}_\eta$ the auctioneer maximizes over $[\eta, C]^{\cdot}$ rather than $[0, C]^{\cdot}$; at a fixed point, $U^{*s}_\eta(i) = \eta$ plays the role that $U^{*s}(i) = 0$ plays on $\mathcal{U}$, and the auctioneer's FOC reads $Z^s \leq 0$ at $U^{*s}_\eta(i) = \eta$, $Z^s = 0$ at interior, and $Z^s \geq 0$ at $U^{*s}_\eta(i) = C$.

and $U_\eta^{*s}(i) = C > 0$, we get $\phi^s = u^s(i, k^{\bar{s}}) - C/\lambda^s$ with $\lambda^s > 0$; substituting into (F′) or directly combining (F) with (B) on side $\bar{s}$,

$$0 \leq \phi^s + \phi^{\bar{s}} \leq \left(u^s(i, k^{\bar{s}}) - C/\lambda^s\right) + u^{\bar{s}}(k^{\bar{s}}, i) = f(i, k^{\bar{s}}) - C/\lambda^s,$$

so $C \leq \lambda^s f(i, k^{\bar{s}}) \leq f(i, k^{\bar{s}}) \leq \max f$, contradicting the choice of $C > \max f$. So $L(k^a, k^b; U_\eta^*) = \emptyset$ at every market with $k^s = i$, hence such markets cannot lie in $\arg\max \hat{V}(\cdot; U_\eta^*)$, contradicting that μ_η^* places positive mass there. Thus $V^*(U_\eta^*) \leq c$ for all sufficiently small η, and by continuity of V^*, $V^*(U^*) \leq c$.

Step 2: Market Clearing. Fix (s, i) and consider the auctioneer's FOC at (U_η^*, μ_η^*) for small η: $Z^s(i; U_\eta^*, \mu_\eta^*) \leq 0$ if $U_\eta^{*s}(i) = \eta$ (lower boundary), $Z^s = 0$ if $U_\eta^{*s}(i) \in (\eta, C)$, and $U_\eta^{*s}(i) = C$ is ruled out by Step 1 at any type with positive demand. Take the limit $\eta \downarrow 0$. If $U^{*s}(i) = 0$, then $U_\eta^{*s}(i) = \eta$ eventually, and $Z^s(i; U_\eta^*, \mu_\eta^*) \leq 0$ passes to the limit because at each η, market clearing gives $\mu_\eta^*(k^a, k^b) \leq \lambda^{s*}(k^a, k^b; U_\eta^*) \cdot I^s F^s(\{i\})$ at every market with $k^s = i$, and $\lambda^{s*}(k^a, k^b; U_\eta^*) \to 0$ along the sequence (by closed graph of $\arg\max$ (29) from Lemma 11 and the fact that at $U^{*s}(i) = 0$ the argmax has $\lambda^{s*} = 0$ on side s); hence $\mu^*(k^a, k^b) = 0$ at every market with $k^s = i$, so $D^s(i; U^*, \mu^*) = 0$ and $Z^s(i; U^*, \mu^*) = -I^s F^s(\{i\}) \leq 0$.[25] If instead $U^{*s}(i) > 0$, then $U_\eta^{*s}(i) \in (\eta, C)$ eventually (since $\eta \downarrow 0$ while $U_\eta^{*s}(i) \to U^{*s}(i) > 0$), so $Z^s(i; U_\eta^*, \mu_\eta^*) = 0$ on the tail, and continuity of D^s on $\mathcal{U}^\circ$ gives $Z^s(i; U^*, \mu^*) = 0$. In either case, Market Clearing (Definition 2) holds: $\sum_{i \in \mathbb{I}'} Z^s(i; U^*, \mu^*) \leq 0$ for any $\mathbb{I}' \subseteq \mathbb{I}^s$, with equality when $U^{*s}(i) > 0$ for all $i \in \mathbb{I}'$.

Step 3: Verifying a CSE. We split into two cases.

Case A: $V^*(U^*) = c$. On the tail, μ_η^* is supported on $A(U_\eta^*)$; upper hemicontinuity of A (established in the first paragraph of the proof) yields $\text{supp}(\mu^*) \subseteq A(U^*)$, so every $(k^a, k^b) \in \text{supp}(\mu^*)$ has $L(k^a, k^b; U^*) \neq \emptyset$ and $\hat{V}(k^a, k^b; U^*) = c$. Since $c > 0$, this forces $m(\lambda^*) > 0$ and $\phi^{a*} + \phi^{b*} > 0$ at every such market. Every $\tau \in \text{supp}(\mu^*)$ is a separating terms-of-trade at (k^a, k^b) with $(\lambda^{a*}, \lambda^{b*}, \phi^{a*}, \phi^{b*}) \in \arg\max$ (29) and value c, so condition (i) of Proposition 1 Part 2 holds. Step 2 gives Market Clearing.

For Equilibrium Utility, we must show $U^{*s}(i) \geq \bar{U}^s(i, \tau, \lambda^{s*})$ for every i and $\tau \in \text{supp}(\mu^*)$. When τ serves $k^s = i$, equality follows from (P). When $k^s > i$, the downward constraint (IC) enforced by $\arg\max$ (29) gives the bound. When $k^s < i$, the bound is an *upward* incentive constraint; we rule out its violation as follows. Suppose $\bar{U}^s(i, \tau, \lambda^{s*}) > U^{*s}(i)$ at some $\tau \in \text{supp}(\mu^*)$ with $k^s < i$. The construction in the proof of Corollary 1—which uses only

[25] The vanishing of μ_η^* at blow-up markets is what prevents the discontinuity of D^s at the boundary from affecting the limit: although μ/λ^{s*} is individually unbounded as $U^s \to 0$, market clearing at each η forces μ_η^* to shrink at the same rate as λ^{s*}, keeping the ratio bounded by supply $I^s F^s(\{i\})$.

Common Ranking, Supermodularity, and the existence of τ itself, and does *not* assume that (U^*, μ^*) is already a CSE—produces a separating terms-of-trade $\tilde{\tau}$ targeting a type $\tilde{k}^s \geq i$ on side s, feasible at U^*, with value strictly exceeding $m(\lambda^*(\tau))(\phi^{a*} + \phi^{b*}) = c$. This contradicts $V^*(U^*) \leq c$ (Step 1), so no such τ exists. Hence condition (ii) holds. Condition (iii) is $V^*(U^*) = c$ by the case assumption. Applying Proposition 1 Part 2, (U^*, μ^*) is a CSE.

Case B: $V^*(U^*) < c$. On the tail, $\mu^*_\eta = 0$ and $Z^s(i; U^*_\eta, 0) = -I^s F^s(\{i\}) \leq 0$, so the auctioneer's optimum on $\mathcal{U}_\eta$ places $U^{*s}_\eta(i) = \eta$ for every (s, i). Hence $U^* = 0$ and $\mu^* = 0$. Take $T = \emptyset$, $\mu^* = 0$, and Λ assigning any $\lambda \in \mathbb{\Lambda}$ on T^p; Free Entry is vacuous, Market Clearing gives $0 \leq$ supply with no positive-utility types to enforce equality, and the Equilibrium Utility condition gives $U^{*s}(i) = \max\{0, \sup_{\tau \in T} \bar{U}^s(i, \tau, \Lambda^s(\tau))\} = 0$, reading the sup over the empty set as $-\infty$. The trivial CSE ($U^* \equiv 0, \mu^* = 0$) is verified directly from Definition 2. ■

F.5 Non-existence under Strict DWTP

Under strict DWTP, the fixed-point argument underlying Theorem 1 fails. This subsection exhibits a two-type, symmetric example satisfying Common Ranking, Supermodularity, and strict DWTP in which no CSE exists for a range of values of c. This example shows that the theorem's restriction to strict IWTP is substantive, not merely a technical convenience.

Types: $\mathbb{I}^a = \mathbb{I}^b = \{0, 1\}$, symmetric payoffs: $u^a(i, j) = u^b(i, j) \equiv u(i, j)$ with

$$u(0,0) = 3, \quad u(0,1) = 5, \quad u(1,0) = 1, \quad u(1,1) = 4.$$

Matching function: $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ on the unit simplex $\mathbb{\Lambda} = \{(\lambda^a, \lambda^b) \geq 0 : \lambda^a + \lambda^b \leq 1\}$, populations: $F^a(\{i\}) = F^b(\{i\}) = 1/2$ with $I^a = I^b = 1$, intermediation cost: $c = 7$. We can verify (strict) Common Ranking, (strict) Supermodularity and strict DWTP directly.

We claim no CSE exists in this example. By Proposition 1, any CSE is separating. Additionally, all feasible terms of trade have $\phi^s \leq u(k^s, k^{\bar{s}})$. Thus a $(0,0)$, $(0,1)$, and $(1,0)$ match all have $\phi^a + \phi^b \leq 6$. Since $m \leq 1$, it follows that $c > m(\lambda^a, \lambda^b)(\phi^a + \phi^b)$ in any terms-of-trade that attracts type 0 agents. Since no such terms of trade can exist, $U^a(0) = U^b(0) = 0$. Since U^s is nonincreasing (Lemma 2), $U^a(1) = U^b(1) = 0$ as well.

Now consider a $(1,1)$ market with $\phi^a = \phi^b = 4$. The incentive and participation constraints are satisfied if and only if $\lambda^s = 0$, giving a platform matching probability of 1 and hence profits of $\phi^a + \phi^b = 8 > 7$. Thus this deviation is profitable, so no CSE exists.

The applications in Section 6 and Section 7.2 of the main text construct DWTP equilibria directly without relying on Theorem 1; the present example does not affect those results.

F.6 Non-existence under Non-Monotone WTP

Monotone WTP cannot be dispensed with in Theorem 1. This subsection exhibits a three-type economy in which Common Ranking and Supermodularity hold, but Monotone WTP fails, and no symmetric CSE exists. The failure traces to the conflicting IC bounds in Step 4: with $u(\cdot, 3)$ first increasing then decreasing across types, the downward ICs from types 1 and 2 onto the $(3, 3)$ market act in opposite directions, and $\hat{V}(3, 3; U)$ is discontinuous as the two bounds cross. This is exactly the mechanism that Monotone WTP is designed to preclude in Lemma 11.

Set $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ on the unit simplex ($\lambda^a, \lambda^b \geq 0$, $\lambda^a + \lambda^b \leq 1$), the parametric matching function of the main text with $\gamma = 1$. Intermediation costs are $c = 1$. Types are labelled $1, 2, 3$; populations and payoffs are symmetric across sides, so $F^a = F^b \equiv F$ and $u^a(i, j) = u^b(j, i) \equiv u(i, j)$. Match payoffs are

$$\big[u(i,j)\big]_{i,j=1,2,3} = \begin{pmatrix} 1.000 & 1.001 & 1.090 \\ 1.090 & 1.100 & 1.210 \\ 0.010 & 0.100 & 1.200 \end{pmatrix}.$$

One can verify by direct computation that this payoff matrix satisfies Common Ranking (column-wise increasing) and strict Supermodularity. It does *not* satisfy Monotone WTP: reading down a column, $u(i, j)$ first increases from $i = 1$ to $i = 2$, then decreases sharply from $i = 2$ to $i = 3$.

By Proposition 1, any CSE is separating, with active markets drawn from the six type pairs $(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)$. We look for a symmetric equilibrium in which $U^a(i) = U^b(i) \equiv U(i)$ and show no such U exists.

Step 1: Lower bounds on $U(1)$ and $U(2)$. Consider an unconstrained $(1, 1)$ market: the value $\hat{V}_1(U(1)) \equiv \max_{(\lambda^a,\lambda^b)\in\Lambda}(1 - \lambda^a - \lambda^b)\big(2u(1, 1) - U(1)/\lambda^a - U(1)/\lambda^b\big)$ is strictly decreasing in $U(1)$. Defining $U(1)^*$ by $\hat{V}_1(U(1)^*) = c = 1$ gives $U(1)^* = \frac{3}{4} - 1/\sqrt{2} \approx 0.0429$. If $U(1) < U(1)^*$ the $(1, 1)$ market earns supernormal profit, inconsistent with CSE; hence $U(1) \geq U(1)^*$.

Similarly, define $U(2)^*$ by setting the unconstrained $(2, 2)$-market value to c; direct computation gives $U(2)^* = \frac{1}{10}(8 - \sqrt{55}) \approx 0.0584$. One verifies that at $U(2)^*$ and any $U(1) \geq U(1)^*$, the downward-incentive constraint $U(1) - U(2)^* \geq \lambda^s(u(1, 2) - u(2, 2))$ is slack at the unconstrained maximizer; so $(2, 2)$-market profit reaches c at $U(2)^*$ and exceeds c at any $U(2) < U(2)^*$, giving $U(2) \geq U(2)^*$.

Step 2: Markets $(1,2)$, $(1,3)$, and $(2,3)$ cannot be active. The unconstrained $(1,2)$-market value at $(U(1)^*, U(2)^*)$ equals

$$\max_{(\lambda^a,\lambda^b)\in\Lambda} (1-\lambda^a-\lambda^b)\big(u(1,2)+u(2,1)-U(1)^*/\lambda^a - U(2)^*/\lambda^b\big) \approx 0.9946 < c.$$

Since the constrained value is weakly lower, and raising $U(1)$ or $U(2)$ further only decreases it, no $(1,2)$ market can be active. Analogous calculations give unconstrained values for $(1,3)$ and $(2,3)$ at $U(3)=0$ of ≈ 0.7085 and ≈ 0.8153 respectively, both below c. Markets $(1,3)$ and $(2,3)$ therefore cannot be active either.

Step 3: Market clearing forces $(1,1),(2,2),(3,3)$ all active. Given that $(1,2),(1,3),(2,3)$ are inactive, type i on either side must match only at the (i,i) market. Market Clearing then requires all three of $(1,1),(2,2),(3,3)$ to be active. Free entry pins $U(1)=U(1)^*$, $U(2)=U(2)^*$.

Step 4: The $(3,3)$ market cannot clear at profit c. At $(U(1)^*, U(2)^*)$ and any $U(3)\geq 0$, the $(3,3)$-market value solves

$$\max_{(\lambda^a,\lambda^b)\in\Lambda} (1-\lambda^a-\lambda^b)\big(2u(3,3)-U(3)/\lambda^a - U(3)/\lambda^b\big)$$

subject to the two downward incentive constraints

$$U(1)^* - U(3) \geq \lambda^s\big(u(1,3)-u(3,3)\big),$$
$$U(2)^* - U(3) \geq \lambda^s\big(u(2,3)-u(3,3)\big).$$

This is where the failure of Monotone WTP bites: the two coefficients on λ^s have *opposite* signs. $u(1,3)-u(3,3)=-0.11<0$, so the first constraint is a lower bound on λ^s; $u(2,3)-u(3,3)=0.01>0$, so the second is an upper bound. Every $U(3)$ above the value 0.0571 at which both constraints bind simultaneously violates one or the other.

At $U(3)=0.0571$, both constraints bind simultaneously at $\lambda^a=\lambda^b\approx 0.1291$ and the $(3,3)$-market value equals $\approx 1.1242 > c$. The market value is strictly decreasing in $U(3)$ (through both the objective and the tightening of the constraints), so at any smaller $U(3)$ the value is at least 1.1242. Thus the $(3,3)$ market earns supernormal profit whenever $(1,1)$ and $(2,2)$ break even, contradicting free entry. This proves no symmetric CSE exists.

Remark on asymmetric CSE. The construction above rules out symmetric CSE. The same mechanism — conflicting IC coefficients at the $(3,3)$ market — obstructs asymmetric

CSE as well, as we have verified by numerical grid search over the CSE-consistent region of (U^a, U^b). A rigorous analytical proof requires tracking which incentive constraints bind at the $(1,2)$, $(2,1)$, and $(2,2)$ markets across the four-dimensional parameter space of type-1 and type-2 utilities, which we find tangential to the purpose of this example. The qualitative lesson — that Monotone WTP cannot be dropped from Theorem 1 — is already established by Steps 1–4.

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Guide to model configurations

Configuration	**Willingness-to-pay**	**Cost**	**Symmetry**	**Location**
PAM	IWTP	Zero	Symmetry	Section 5
Anchor	IWTP	Zero	Symmetry	Section 5
PAM	DWTP	Zero	Symmetry	Section 6
Anchor	DWTP	Zero	Symmetry	Section 6
PAM	IWTP/DWTP	Zero	Asymmetry	Section 7.1
PAM	IWTP/DWTP	Positive	Symmetry	Section 7.2
NAM	IWTP/DWTP	Zero	Symmetry	Online Appendix G
PAM	IWTP/DWTP	Positive	Asymmetry	Online Appendix H

G Negative Sorting with Zero Costs and Symmetry

This section characterizes negative assortative matching (NAM) CSE in a symmetric environment with zero intermediation costs, for both IWTP and DWTP. Characterization 2 gives the differential equation system for the IWTP case, and Characterization 3 does the same for DWTP. We illustrate the IWTP case with a numerical example in Figure 8, where we also verify global incentive constraints and the absence of profitable platform deviations.

Characterization 2 (NAM with Symmetry and Zero Costs under IWTP) *Assume Common Ranking, Supermodularity, Limit Supermodularity, Symmetric Environment, and IWTP. Also assume $c = 0$. In a negative sorted equilibrium, for the terms-of-trade attracting types $(i, \sigma(i))$, denote the side-a and side-b matching probabilities as $\ell^a(i)$ and $\ell^b(i)$. The following equation system characterizes the equilibrium outcomes:*

$$
\begin{gathered}
m(\ell^a(i), \ell^b(i)) = 0, \\
U'(i) = \ell^a(i)\, u_1(i, \sigma(i)), \qquad U'(\sigma(i)) = \ell^b(i)\, u_1(\sigma(i), i), \\
U(i) = \ell^a(i)\, (u(i, \sigma(i)) - \Phi^a(i)), \qquad U(\sigma(i)) = \ell^b(i)\, (u(\sigma(i), i) - \Phi^b(i)), \\
\frac{\ell^{a\prime}(i)}{\ell^a(i)} + \frac{u_{12}(i, \sigma(i))}{u_1(i, \sigma(i))}\sigma'(i) = \frac{m_2(\ell^a(i), \ell^b(i))\, \ell^b(i)}{m_1(\ell^a(i), \ell^b(i))\, \ell^a(i)}\, \frac{u_{12}(\sigma(i), i)}{u_1(\sigma(i), i)}, \\
\sigma'(i) = -\frac{f(i)\, \ell^a(i)}{f(\sigma(i))\, \ell^b(i)},
\end{gathered}
$$

with boundary conditions $\sigma(\underline{i}) = \bar{i}$, $\sigma(\bar{i}) = \underline{i}$, and $\Phi^a(\underline{i}) + \Phi^b(\bar{i}) = 0$ in the $(\underline{i}, \bar{i})$ market.

The system parallels Characterization 1 for PAM with zero costs, with two differences. First, σ is decreasing, so σ' has a negative sign. Second, the boundary conditions pair the lowest side-a type with the highest side-b type (and vice versa). As in the PAM case, the zero-fee condition at the $(\underline{i}, \bar{i})$ market pins down the level of U; after this computation, one still has to verify global incentive constraints and the absence of profitable platform deviations, which we do numerically in the example below.

Figure 8 illustrates the equilibrium and compares it to the observable-types benchmark. Many patterns are similar across the two environments: higher types match with lower types, higher types match with a higher probability, and lower types pay a higher fee. The platform matching probability is zero throughout. The main difference is that the sum of fees collected is zero with observable types but strictly positive with private information, except in the $(\underline{i}, \bar{i})$ market.

Proof of Characterization 2. The derivation parallels the proof of Characterization 1; we indicate the points of difference.

Zero platform matching probability and local IC. Lemma 4 gives $m(\ell^a(i), \ell^b(i)) = 0$, the first equation. For a platform attracting (i, j), the local ICs give $\lambda^a(i,j) = U'(i)/u_1(i,j)$ and $\lambda^b(i,j) = U'(j)/u_1(j,i)$, the second line. The third line is the participation constraint in this same market.

Platform optimality. Writing $\hat{V}(i,j) = m(\lambda^a(i,j), \lambda^b(i,j)) \cdot S(i,j)$ with S the sum of fees, and using $m(\ell^a(i), \ell^b(i)) = 0$ as before, the FOC $\hat{V}_1(i, \sigma(i)) = 0$ reduces to

$$0 = m_1(\ell^a(i), \ell^b(i)) \left.\frac{\partial \lambda^a}{\partial i}\right|_{j=\sigma(i)} + m_2(\ell^a(i), \ell^b(i)) \left.\frac{\partial \lambda^b}{\partial i}\right|_{j=\sigma(i)}. \tag{31}$$

The same calculations as in the proof of Characterization 1 give

$$\left.\frac{\partial \lambda^b}{\partial i}\right|_{j=\sigma(i)} = -\ell^b(i) \frac{u_{12}(\sigma(i), i)}{u_1(\sigma(i), i)}, \qquad \left.\frac{\partial \lambda^a}{\partial i}\right|_{j=\sigma(i)} = \ell^{a\prime}(i) + \ell^a(i) \frac{u_{12}(i, \sigma(i))}{u_1(i, \sigma(i))} \sigma'(i).$$

Substituting into (31) and dividing by $m_1(\ell^a(i), \ell^b(i))\, \ell^a(i)$ yields the fourth line in the characterization.

Market clearing. In NAM, side-a type i matches with side-b type $\sigma(i)$, where σ is decreasing. Consistency of the equilibrium measures gives $d\nu^b/d\nu^a = \ell^a/\ell^b$ on the active curve. Equating accumulated side-b agents used by side-a types in $[\underline{i}, i]$ with available supply in $[\sigma(i), \bar{i}]$:

$$I^b\big(F(\bar{i}) - F(\sigma(i))\big) = \int_{\underline{i}}^{i} \frac{\ell^a(x)}{\ell^b(x)}\, I^a \, dF(x).$$

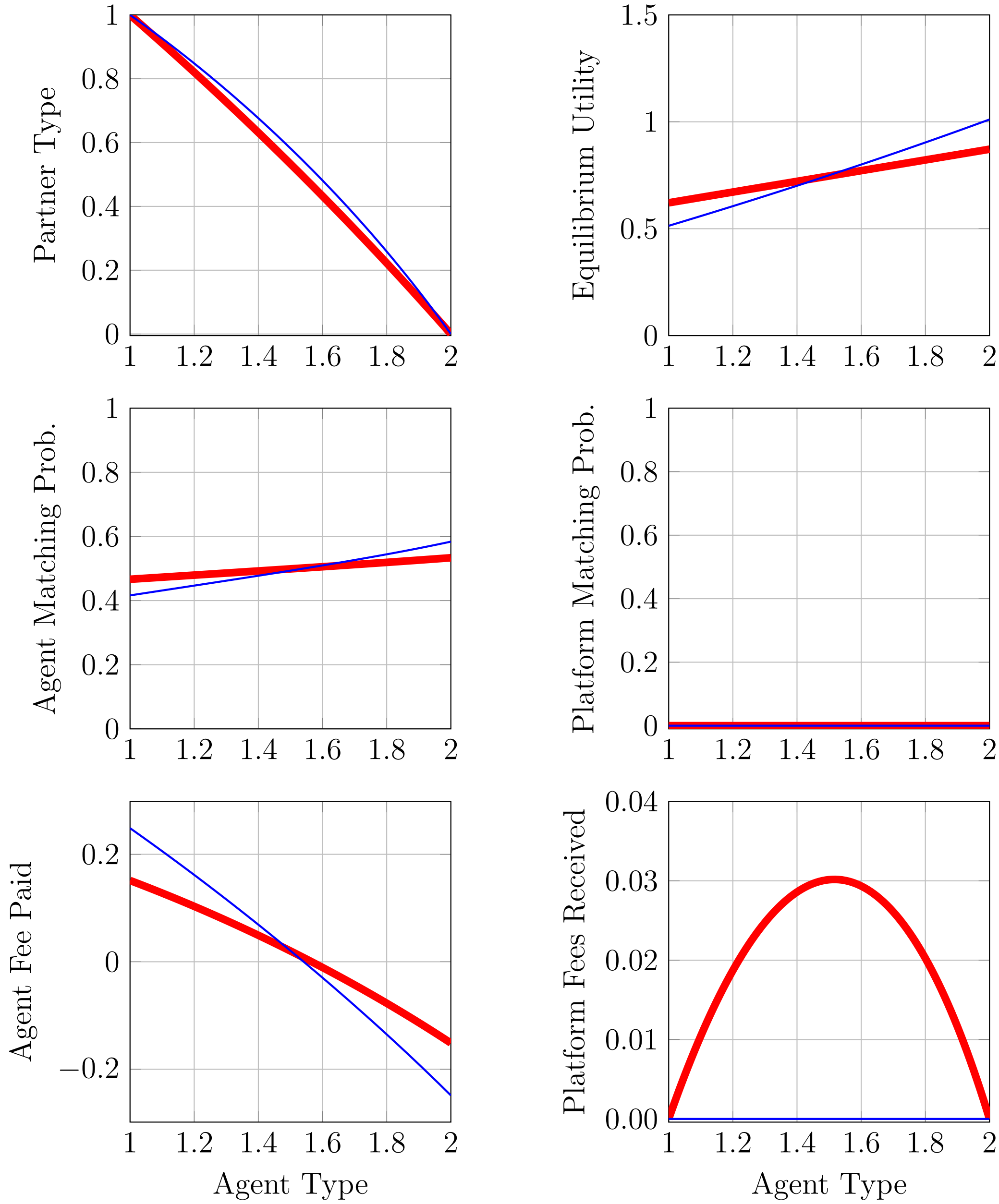

Figure 8: Negative Assortative Matching with IWTP. Notes: Thick red lines represent private-information equilibrium outcomes, while thinner blue lines show observable-type equilibrium outcomes. Payoff function $u(i,j) = (0.5i^{0.8} + 0.5j^{0.8})^{1.25}$ ($\theta = 5$), matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ ($\gamma = 1$), types are distributed uniformly on $[1, 2]$.

Under Symmetric Environment, $I^a = I^b$ and the two sides share the distribution F. Differentiating and solving:

$$\sigma'(i) = -\frac{f(i)\,\ell^a(i)}{f(\sigma(i))\,\ell^b(i)} < 0,$$

the fifth line in the characterization.

Boundary conditions. Market clearing at the endpoints requires $\sigma(\underline{i}) = \bar{i}$ and $\sigma(\bar{i}) = \underline{i}$. In the $(\underline{i}, \bar{i})$ market, type $\underline{i}$ on side a faces no downward ICs, so λ^a is unconstrained, while λ^b must satisfy the local IC for type $\bar{i}$. The zero-fee condition $\Phi^a(\underline{i}) + \Phi^b(\bar{i}) = 0$ pins down the level of U. ■

Characterization 3 (NAM with Symmetry and Zero Costs under DWTP) *Assume Common Ranking, Supermodularity, Limit Supermodularity, Symmetric Environment, and DWTP. Also assume $c = 0$. In a negative sorted equilibrium, for the terms-of-trade attracting types $(i, \sigma(i))$, denote the side-a and side-b matching probabilities as $\ell^a(i)$ and $\ell^b(i)$. The following equation system characterizes the equilibrium outcomes:*

$$\begin{gathered}
u(i,\sigma(i)) + u(\sigma(i),i) - \frac{U(i)}{\ell^a(i)} - \frac{U(\sigma(i))}{\ell^b(i)} = 0,\\
U'(i) = \ell^a(i)\,u_1(i,\sigma(i)), \qquad U'(\sigma(i)) = \ell^b(i)\,u_1(\sigma(i),i),\\
U(i) = \ell^a(i)\,(u(i,\sigma(i)) - \Phi^a(i)), \qquad U(\sigma(i)) = \ell^b(i)\,(u(\sigma(i),i) - \Phi^b(i)),\\
u_2(\sigma(i),i) + \frac{U(i)}{\ell^a(i)}\left(\frac{\ell^{a\prime}(i)}{\ell^a(i)} + \frac{u_{12}(i,\sigma(i))}{u_1(i,\sigma(i))}\sigma'(i)\right) - \frac{U(\sigma(i))}{\ell^b(i)}\,\frac{u_{12}(\sigma(i),i)}{u_1(\sigma(i),i)} = 0,\\
\sigma'(i) = -\frac{f(i)\,\ell^a(i)}{f(\sigma(i))\,\ell^b(i)},
\end{gathered}$$

with boundary conditions $\sigma(\underline{i}) = \bar{i}$, $\sigma(\bar{i}) = \underline{i}$, and $m(\ell^a(\underline{i}), \ell^b(\underline{i})) = 0$ in the $(\underline{i}, \bar{i})$ market.

The DWTP case differs structurally from the IWTP case: with DWTP the platform matching probability is generically positive, while the sum of fees is zero in every active market. The first equation replaces $m = 0$ (which held under IWTP) with $\phi^a + \phi^b = 0$. The remaining equations are the local ICs, participation constraints, platform's partner-choice FOC, and market clearing, derived as in the IWTP case.

Proof of Characterization 3. The local ICs, participation constraints, and market-clearing derivation are identical to those in Characterization 2. With DWTP the platform matching probability is positive; instead the sum of fees is zero, i.e., $S(i, \sigma(i)) = 0$, where S is the fee sum from equation (10). This gives the first equation in the characterization.

Writing $\hat{V} = m \cdot S$, the FOC $\hat{V}_1(i, \sigma(i)) = 0$ becomes $m(\ell^a(i), \ell^b(i)) \cdot S_1(i, \sigma(i)) = 0$ (the $m_1 S$ and $m_2 S$ terms vanish because $S = 0$). Assuming $m(\ell^a(i), \ell^b(i)) \neq 0$, we require $S_1(i, \sigma(i)) = 0$. Differentiating S and using the local IC to cancel the $u_1(i, \sigma(i))$ term,

$$S_1(i, \sigma(i)) = u_2(\sigma(i), i) + \frac{U(i)}{(\ell^a(i))^2} \frac{\partial \lambda^a}{\partial i}\bigg|_{j=\sigma(i)} + \frac{U(\sigma(i))}{(\ell^b(i))^2} \frac{\partial \lambda^b}{\partial i}\bigg|_{j=\sigma(i)}.$$

Substituting the expressions for $\partial\lambda^a/\partial i$ and $\partial\lambda^b/\partial i$ from the proof of Characterization 2 yields the fourth equation in the characterization.

Finally, the boundary condition $m(\ell^a(\underline{i}), \ell^b(\underline{i})) = 0$ at the $(\underline{i}, \bar{i})$ market arises from the unconstrained problem for type $\underline{i}$ on side a. This pushes λ^a to the boundary of the feasible set, where $m = 0$. ■

H Equilibrium with Positive Costs and Asymmetry

This section characterizes CSE with positive costs and asymmetry between the two sides, generalizing the results in Section 7.1 (which assumed zero costs) and in Section 7.2 (which assumed symmetry). Section H.1 treats PAM, Section H.2 treats NAM. As in Section 7.2, the differential equations characterize a CSE whenever one exists; verifying existence requires checking the platform value $\hat{V}(k^a, k^b)$ in equation (10) for all (k^a, k^b), which is a numerical exercise.

H.1 Positive Assortative Matching

For expositional simplicity, we focus on equilibria where every type participates; *Discussion* at the end of this subsection addresses the possibility of non-participation. We denote by $[\underline{i}^s, \bar{i}^s]$ the support of the type distribution on side s, with density f^s.

Characterization 4 (PAM with Asymmetry and Positive Costs) *Assume Common Ranking and Supermodularity. In a PAM CSE with all types participating, for the terms-of-trade attracting types $(i, \sigma(i))$, denote the side-s matching probability as $\ell^s(i)$ and equilibrium* *utility as U^s. The following equation system characterizes the equilibrium outcomes:*

$$\xi^a(i)\left(\frac{\ell^{a\prime}(i)}{\ell^a(i)}+\frac{u^a_{12}(i,\sigma(i))}{u^a_1(i,\sigma(i))}\sigma'(i)\right)-\xi^b(i)\frac{u^b_{12}(\sigma(i),i)}{u^b_1(\sigma(i),i)}+u^b_2(\sigma(i),i)=0,$$
$$\xi^b(i)\frac{1}{\sigma'(i)}\left(\frac{\ell^{b\prime}(i)}{\ell^b(i)}+\frac{u^b_{12}(\sigma(i),i)}{u^b_1(\sigma(i),i)}\right)-\xi^a(i)\frac{u^a_{12}(i,\sigma(i))}{u^a_1(i,\sigma(i))}+u^a_2(i,\sigma(i))=0,$$
$$U^{a\prime}(i)=\ell^a(i)\,u^a_1(i,\sigma(i)),\qquad U^{b\prime}(\sigma(i))=\ell^b(i)\,u^b_1(\sigma(i),i),$$
$$U^a(i)=\ell^a(i)\,(u^a(i,\sigma(i))-\Phi^a(i)),\qquad U^b(\sigma(i))=\ell^b(i)\,(u^b(\sigma(i),i)-\Phi^b(i)),$$
$$\sigma'(i)=\frac{I^a f^a(i)\,\ell^a(i)}{I^b f^b(\sigma(i))\,\ell^b(i)},$$

with boundary conditions $\sigma(\underline{i}^a)=\underline{i}^b$ and $\sigma(\bar{i}^a)=\bar{i}^b$, together with the condition that, for any fixed $U^a(\underline{i}^a)$, the tuple $(\ell^a(\underline{i}^a),\ell^b(\underline{i}^a),U^b(\underline{i}^b))$ solves the unconstrained problem for the $(\underline{i}^a,\underline{i}^b)$ market:

$$c=\max_{\lambda^a,\lambda^b} m(\lambda^a,\lambda^b)\left(u^a(\underline{i}^a,\underline{i}^b)+u^b(\underline{i}^b,\underline{i}^a)-\frac{U^a(\underline{i}^a)}{\lambda^a}-\frac{U^b(\underline{i}^b)}{\lambda^b}\right). \tag{32}$$

Here

$$\xi^s(i)\equiv\frac{\epsilon_s(\ell^a(i),\ell^b(i))}{m(\ell^a(i),\ell^b(i))}c+\frac{U^s(k^s)}{\ell^s(i)},\quad \epsilon_s(\lambda^a,\lambda^b)\equiv\frac{\lambda^s\,m_s(\lambda^a,\lambda^b)}{m(\lambda^a,\lambda^b)},$$

with $k^a=i$, $k^b=\sigma(i)$.

Proof of Characterization 4. The derivation parallels the proof of Characterization 1, with two main differences. First, with $c>0$ the platform matching probability m is generically nonzero, so the FOC for platform optimality picks up a term from $m\cdot S_1$ that was absent in the zero-cost case. Second, with asymmetry we obtain two FOCs (one from varying each side's type), yielding the first two lines in the characterization.

Local IC and market clearing. The local IC (Lemma 3) gives $\lambda^a(i,j)=U^{a\prime}(i)/u^a_1(i,j)$, $\lambda^b(i,j)=U^{b\prime}(j)/u^b_1(j,i)$, which is the third line. Market clearing gives the fifth line exactly as in the proof of Characterization 1. The fourth line is the participation constraint.

Platform optimality. Write $\hat V(i,j)=m(\lambda^a(i,j),\lambda^b(i,j))\cdot S(i,j)$, with $S(i,j)\equiv u^a(i,j)+u^b(j,i)-U^a(i)/\lambda^a(i,j)-U^b(j)/\lambda^b(i,j)$. Along the equilibrium path, $m\cdot S=c$, so $S=c/m$. Platform optimality gives $\hat V_1(i,\sigma(i))=0$. Applying the product rule:

$$\hat V_1=\left(m_1\frac{\partial\lambda^a}{\partial i}+m_2\frac{\partial\lambda^b}{\partial i}\right)S+m\,S_1.$$

Using the local IC to cancel the u_1^a term in S_1,

$$S_1\Big|_{j=\sigma(i)} = u_2^b(\sigma(i), i) + \frac{U^a(i)}{(\ell^a(i))^2}\frac{\partial \lambda^a}{\partial i}\Big|_{j=\sigma(i)} + \frac{U^b(\sigma(i))}{(\ell^b(i))^2}\frac{\partial \lambda^b}{\partial i}\Big|_{j=\sigma(i)}.$$

Substituting $S = c/m$ and collecting terms yields

$$\hat{V}_1 = \left(\frac{m_1\, c}{m} + \frac{m\, U^a(i)}{(\ell^a(i))^2}\right)\frac{\partial \lambda^a}{\partial i} + \left(\frac{m_2\, c}{m} + \frac{m\, U^b(\sigma(i))}{(\ell^b(i))^2}\right)\frac{\partial \lambda^b}{\partial i} + m\, u_2^b(\sigma(i), i).$$

Using the definitions of ξ^s and ϵ_s, each bracketed coefficient equals $m\,\xi^s(i)/\ell^s(i)$. Dividing through by m:

$$0 = \frac{\xi^a(i)}{\ell^a(i)}\frac{\partial \lambda^a}{\partial i}\Big|_{j=\sigma(i)} + \frac{\xi^b(i)}{\ell^b(i)}\frac{\partial \lambda^b}{\partial i}\Big|_{j=\sigma(i)} + u_2^b(\sigma(i), i).$$

The partial derivatives $\partial\lambda^a/\partial i$ and $\partial\lambda^b/\partial i$, evaluated at $j = \sigma(i)$, are the same as in the proof of Characterization 1:

$$\frac{\partial \lambda^a}{\partial i}\Big|_{j=\sigma(i)} = \ell^{a\prime}(i) + \ell^a(i)\frac{u_{12}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))}\sigma'(i), \qquad \frac{\partial \lambda^b}{\partial i}\Big|_{j=\sigma(i)} = -\ell^b(i)\frac{u_{12}^b(\sigma(i), i)}{u_1^b(\sigma(i), i)}.$$

Substituting yields the first line of the characterization.

The analogous FOC $\hat{V}_2(i, \sigma(i)) = 0$, obtained by varying j with i fixed, yields the second line by symmetric computations, using

$$\frac{\partial \lambda^a}{\partial j}\Big|_{j=\sigma(i)} = -\ell^a(i)\frac{u_{12}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))}, \qquad \frac{\partial \lambda^b}{\partial j}\Big|_{j=\sigma(i)} = \frac{1}{\sigma'(i)}\left(\ell^{b\prime}(i) + \ell^b(i)\frac{u_{12}^b(\sigma(i), i)}{u_1^b(\sigma(i), i)}\right),$$

where the second expression is derived by totally differentiating the identity $\ell^b(i) = \lambda^b(i, \sigma(i))$.

Boundary conditions. Market clearing at the endpoints requires $\sigma(\underline{i}^a) = \underline{i}^b$ and $\sigma(\bar{i}^a) = \bar{i}^b$. The $(\underline{i}^a, \underline{i}^b)$ market is unconstrained (Proposition 1), giving problem (32). ■

Discussion. If the differential equation system in Characterization 4 has no solution with $U^s(i) \geq 0$ for all i, then the PAM equilibrium involves non-participation by some types. Under IWTP, three scenarios are possible: low types on side a do not participate (there exists $i_*^a \in (\underline{i}^a, \bar{i}^a]$ with $U^a(i) = 0$ for $i \leq i_*^a$, and the assignment function starts from i_*^a); the analogous situation holds on side b; or the lowest types on both sides do not participate, which occurs when no terms-of-trade attracting side-s type-$\underline{i}^s$ covers the intermediation cost.

H.2 Negative Assortative Matching with Positive Costs

Our final example is a NAM CSE with positive costs. Its characterization is close to that of a PAM CSE.

Characterization 5 (NAM with Asymmetry and Positive Costs) *Assume Common Ranking and Supermodularity. In a NAM CSE with all types participating, for the terms-of-trade attracting types $(i, \sigma(i))$, denote the side-s matching probability as $\ell^s(i)$ and equilibrium utility as U^s. The equation system in the first four lines of Characterization 4 characterizes the equilibrium outcomes, together with*

$$\sigma'(i) = -\frac{I^a f^a(i)\, \ell^a(i)}{I^b f^b(\sigma(i))\, \ell^b(i)},$$

with boundary conditions $\sigma(\underline{i}^a) = \bar{i}^b$ and $\sigma(\bar{i}^a) = \underline{i}^b$, and the condition that, for any fixed $U^a(\underline{i}^a)$ and $U^b(\underline{i}^b)$, the pairs $(\ell^a(\underline{i}^a), U^b(\bar{i}^b))$ and $(\ell^a(\bar{i}^a), U^b(\underline{i}^b))$ solve the unconstrained problems for the $(\underline{i}^a, \bar{i}^b)$ and $(\bar{i}^a, \underline{i}^b)$ markets, respectively:

$$c = \max_{\lambda^a} m(\lambda^a, \ell^b(\underline{i}^a)) \left(u^a(\underline{i}^a, \bar{i}^b) + u^b(\bar{i}^b, \underline{i}^a) - \frac{U^a(\underline{i}^a)}{\lambda^a} - \frac{U^b(\bar{i}^b)}{\ell^b(\underline{i}^a)} \right), \tag{33}$$

$$c = \max_{\lambda^b} m(\ell^a(\bar{i}^a), \lambda^b) \left(u^a(\bar{i}^a, \underline{i}^b) + u^b(\underline{i}^b, \bar{i}^a) - \frac{U^a(\bar{i}^a)}{\ell^a(\bar{i}^a)} - \frac{U^b(\underline{i}^b)}{\lambda^b} \right), \tag{34}$$

where $\ell^b(\underline{i}^a)$ and $\ell^a(\bar{i}^a)$ are pinned down by the local ICs for types $\bar{i}^b$ and $\bar{i}^a$, respectively.

Proof of Characterization 5. The first four lines follow from the same steps as in the proof of Characterization 4; the derivation does not use the sign of σ'. For market clearing, consistency of the equilibrium measures gives $d\nu^b/d\nu^a = \ell^a/\ell^b$ on the active curve. Equating accumulated side-b agents used by side-a types in $[\underline{i}^a, i]$ with available supply in $[\sigma(i), \bar{i}^b]$:

$$I^b\big(F^b(\bar{i}^b) - F^b(\sigma(i))\big) = \int_{\underline{i}^a}^{i} \frac{\ell^a(x)}{\ell^b(x)}\, I^a\, dF^a(x).$$

Differentiating yields the market-clearing equation. ■