Competitive Sorting with Bilateral Private Information*

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Abstract

Becker (1973) showed how competitive markets allocate partners when types are observable. Akerlof (1970) showed how markets organize trade when one side has private information about its own type. The natural intersection, competitive markets where individuals on both sides have private information about types that matter to their partners, has remained largely unanalyzed. We develop a framework for this intersection. Platforms post fees and announce participant distributions; agents direct their search to the terms-of-trade they prefer, and in equilibrium the announcements are self-fulfilling. Under standard monotonicity and complementarity conditions, equilibrium is separating and only downward incentive constraints bind. How separation operates depends on willingness-to-pay: when higher types value matching more, platforms use fees to screen; when they value it less, platforms use rationing. We also uncover anchor matching, in which low types pair with substantially higher-type partners and earn information rents above the full-information benchmark.

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1 Introduction

Becker (1973) showed how competitive markets allocate partners when types are observable. Akerlof (1970) showed how markets organize trade when one side has private information about its own type. Each has spawned a vast literature. But the natural intersection, competitive markets in which agents on both sides have partner-relevant private information, has remained largely unanalyzed.[1] Workers care about the quality of jobs, while firms care about workers' skill. Dating app users care about hidden attributes of potential matches. Asset sellers care about buyers' expertise in valuation, while buyers care about sellers' information about quality. People care about the risk of getting sick from social interactions, but cannot tell whether others are already infected. In each case, both sides hold payoff-relevant private information, and competition between intermediaries determines the terms on which they meet. We develop a competitive framework for such markets.

A competitive equilibrium has to specify who matches with whom and with what probability, as well as what payments are made. Moreover, it must recognize that any agent can pretend to be any other type if they are willing to accept the matching distributions and payments of that type. And because agents care directly about whom they match with, the incentive to deviate depends on the choices of potential matching partners. Thus sorting environments with private information have a fixed-point problem at their core.

We solve this problem with a framework that works as follows. Homogeneous platforms post terms-of-trade, which consist of fees and distributions of agents on each side of the market. Privately informed heterogeneous agents go to the platform offering them the highest expected utility. Platforms have no informational advantage; their role is to coordinate agents' expectations about who will be where through the terms-of-trade. The terms-of-trade must be consistent with agents' equilibrium behavior: every type the platform claims to attract must prefer it to the alternatives, given the fees and types it claims to attract on the other side. Free entry drives platform profits to zero. Our framework extends the competitive search environment of Eeckhout and Kircher (2010), a natural frictional analog of Becker (1973), to settings with two-sided partner-relevant private information.

A first application of our framework establishes conditions under which any equilibrium must be separating, so each agent knows exactly what payoff they will receive from matching at any terms-of-trade they go to. We assume throughout that all agents agree that higher types are more desirable (Common Ranking). This gives agents an incentive to infiltrate a terms-of-trade that matches them with a high type. Still, we prove that if high types have a stronger preference for higher-type partners (Supermodularity), any terms-of-trade

[1] We discuss Myerson and Satterthwaite (1983) and the bilateral-trade literature in the literature review.

that maximizes platform profits is separating (Proposition 1). Moreover, only downward incentive constraints bind: a platform may need to distort the terms-of-trade in order to thwart lower agent types' desire to mimic higher ones, but in equilibrium higher agent types are not tempted to come to the platform (Corollary 1). These results emerge from competitive pressure between platforms: any attempt to pool types would be vulnerable to rival platforms cream-skimming the higher types. We give an example which shows that separation may fail without Supermodularity.

Much of our analysis focuses on positive assortative matching, where higher-type agents match with higher-type partners. We prove that how platforms attract the desired type crucially depends on whether the willingness-to-pay for a match is increasing or decreasing in an agent's own type. With increasing willingness-to-pay, where higher types value any given match more than lower types, low types are deterred from platforms intended for high types through fees that the low types find prohibitively expensive. For instance, in labor markets, a high-skill worker may value matching with any given firm more than a low-skill worker does. Higher types pay more to access platforms that attract better partners, while all types match at the maximum feasible rate (Proposition 2).

With decreasing willingness-to-pay, where higher types value any given match less than lower types, low types do not go to platforms intended for high types because the matching probability is low. This is a cost that higher types are willing to bear but lower types are not. For example, in disease transmission models, healthy individuals value any given match less than sick individuals do, because matching comes with a risk of becoming sick. Higher (healthier) types match less frequently to avoid undesirable partners, while fees are zero (Proposition 3).

These results—fees alone with increasing willingness-to-pay, matching probabilities alone with decreasing willingness-to-pay—characterize the limiting case where platforms can costlessly facilitate matching. When platforms face positive intermediation costs, both mechanisms operate simultaneously (Proposition 4). With increasing willingness-to-pay, higher types pay higher fees and match more frequently. With decreasing willingness-to-pay, higher types pay lower fees and match less frequently. These patterns reflect the forces in the previous paragraphs, along with the need to cover costs: with increasing willingness-to-pay, platforms charge high types more and so are able to provide higher matching rates. With decreasing willingness-to-pay, high types accept low matching rates to avoid bad partners. Since low matching rates are cheaper for platforms to provide, competition drives down fees.

The conditions needed for a positive assortative matching equilibrium quite generally fail for the lowest types. We show how a richer equilibrium structure may emerge: *anchor matching*. The name is both economic and visual. Economically, some type i^* serves as an anchor: at i^*, equilibrium utility equals the full-information benchmark, and this value ties down the entire equilibrium utility profile through local incentive constraints. Above i^*, the profile looks like a standard positive assortative matching equilibrium with bilateral private information and lowest type i^*. A first-order approach delivers equilibrium utility from the boundary condition at i^*. Below i^*, the profile is determined by a different mechanism we describe below. Visually, the resulting matching pattern takes the shape of an anchor (Figures 2 and 4): a positively sloped shank along the 45-degree line above i^*, with arms that curve outward below i^* to pair the lowest types with substantially higher-type partners.

The central question is what determines i^*. In a standard positive assortative matching equilibrium, $i^* = \underline{i}$: the lowest type is the anchor, its incentive constraints are slack, and its utility equals the full-information benchmark. Anchor matching arises when this value is too low to survive competition. The lowest type faces no downward incentive constraints and so platforms find it profitable to pair them with higher-type partners, bidding up their equilibrium utility. But raising the equilibrium utility of the lowest type relaxes downward incentive constraints on slightly higher types, making them more attractive partners as well and bidding up their equilibrium utility. The anchor i^* is the lowest type for which this profit opportunity has been competed away, or equivalently, the lowest type that sits at the full-information benchmark. Below i^*, equilibrium utility lies above the full-information benchmark: competition delivers a utility level to the lowest types that they could not obtain under observability. Above i^*, the standard private-information penalty applies.

No platform transfers surplus across types: each breaks even. Instead, we recover something that looks like cross-subsidization, with low types receiving information rents beyond the full-information benchmark, through a change in who matches with whom. This mechanism is distinct from the cross-subsidies that fail in Rothschild and Stiglitz (1976): there is no pooling, no deviation to cream-skimming contracts is profitable, and no individual platform runs a loss on any type.

Our results illuminate how real-world markets successfully separate types despite private information. In practice, both fees and matching frequencies are utilized. Building on early insights of Veblen (1900), research by Spence (1973), Pesendorfer (1995), and Bagwell and Bernheim (1996) showed how wasteful expenditures can signal an individual's unobserved qualities. Conversely, research by Guerrieri and Shimer (2014) shows that illiquidity can serve a similar function, with the owners of high-quality assets setting a high price that attracts few buyers. This separates them from the owners of lower-quality assets because the high-quality owners are more willing to accept the risk of not selling the asset.

Our theory predicts when we should see wasteful expenditures (high fees paid to platforms) versus illiquidity (low matching probabilities) in real-world markets. The form separation takes hinges on whether higher types are more or less eager to match overall: under increasing willingness-to-pay, fees screen out low types; under decreasing willingness-to-pay, rationing does.

Online dating markets illustrate the question concretely. Tinder offers a hierarchy of paid tiers, including a "Select" membership at $499 a month with modest additional features. In our framework, paying the Select fee is the visible act of joining a particular terms-of-trade, and the model treats this as the central screening device.[2] Which people pay for Select turns on whether the most desirable partners are also the ones who are the most eager to match (increasing willingness-to-pay) or the least eager (decreasing willingness-to-pay). Under the former assumption, willingness-to-pay tracks desirability: fees screen out low types, and Select sustains a separating equilibrium for the high end users. Under decreasing willingness-to-pay, for example if the most desirable daters use the app casually, willingness-to-pay diverges from desirability, fees fail to screen, and a high-fee tier attracts adversely selected users. Our framework predicts that high-fee terms-of-trade like Tinder Select will only be successful in environments with increasing willingness-to-pay.

Related Literature

Our paper extends the competitive search framework to environments with two-sided private information where agents care about partner types. Early competitive search models had neither heterogeneity nor private information (Moen, 1997; Shimer, 1996). Subsequent research has extended the basic model to allow for two-sided heterogeneity in order to study assortative matching (Shi, 2001; Shimer, 2005; Eeckhout and Kircher, 2010), but still assumes that there is no relevant private information.[3]

Guerrieri, Shimer and Wright (2010) and Guerrieri and Shimer (2014) introduced private information to competitive search but maintained one-sided private information.[4] For example, workers have private information but firms do not, or buyers have private information about their valuation but sellers care only about price. The crucial difference in our environment is that both sides have private information and both sides care about their partner's type. This creates a fundamentally different screening problem. In Guerrieri, Shimer and

[2]Our framework has homogeneous platforms posting terms-of-trade, so whether one firm offers many tiers or many firms each offer one is irrelevant to equilibrium. Competition between Tinder and rival dating platforms disciplines which tiers can survive.

[3]In Eeckhout and Kircher (2010), buyers are privately informed about their type, but sellers do not care about the buyer's type, and so this has no impact on the equilibrium allocation or prices.

[4]Other papers that study environments with one-sided private information and search frictions include Daley and Green (2012); Kurlat (2013); Kim and Kircher (2015); Chang (2018); Auster and Gottardi (2019); Lester, Shourideh, Venkateswaran and Zetlin-Jones (2019, 2023); Albrecht, Cai, Gautier and Vroman (2024); Auster, Gottardi and Wolthoff (2025).

Wright (2010), separation occurs when the uninformed party posts a contract to attract a particular type of informed party. In our model, both sides simultaneously screen and are screened, creating a fixed-point problem in beliefs about who participates in each market.

The literature on matching with two-sided private information has taken different approaches. Hoppe, Moldovanu and Sela (2009) study assortative matching when agents send costly signals, under the assumption that there is positive assortative matching in signals. Their single-crossing conditions ensure higher types send higher signals in equilibrium. Damiano and Li (2007) and Hoppe, Moldovanu and Ozdenoren (2011) analyze monopolistic platforms that screen privately informed agents and Damiano and Li (2008) consider a screening model with duopolistic platforms. Our paper differs in three key ways. First, we develop the natural benchmark of competitive markets, focusing on how agents gain from trade, rather than on how monopolists extract profits. Second, we endogenize both fees and matching rates as screening instruments, showing that separation can work through either channel. Third, we allow general payoff functions, showing that increasing versus decreasing willingness-to-pay fundamentally changes how markets operate.

A natural comparison is the bilateral-trade literature initiated by Myerson and Satterthwaite (1983), which studies trade with two-sided private information. Two features distinguish that environment from ours. First, Myerson-Satterthwaite analyze a single buyer and a single seller. One could embed their bilateral mechanism in a market with many participants, but doing so reduces to a competitive market with adverse selection of the kind we discuss in the next paragraph. This is because of the second, more fundamental feature: agents in their setting care about each other only through the price. A buyer's payoff depends on the seller's private valuation only because that valuation determines whether trade happens, not because the seller's type enters the buyer's utility directly. The same is true in reverse for the seller. We are interested in markets with partner-relevant private information, where the worker's payoff depends on which firm employs them, the dating app user's payoff depends on the character of their match, and the asset buyer's payoff depends on the seller's expertise. This direct dependence on partner type is what makes the matching problem interesting, and it is exactly what Myerson-Satterthwaite, by design, abstracts from.

More recently, Azevedo and Gottlieb (2017) develop a perfectly competitive model of markets with adverse selection in which contract characteristics are endogenously determined. Their setting shares our focus on competitive markets, but they assume, as in the bilateral-trade tradition, that counterparties matter only through the terms-of-trade, not directly through preferences. Together with our paper, the two provide competitive benchmarks for two distinct strands of the private-information literature: theirs for the terms-of-trade-only case, ours for markets with partner-relevant private information.

Our technical approach connects to the mechanism design literature with type-dependent outside options. Lewis and Sappington (1989) show that when types differ in their outside options, incentive constraints can bind in both directions, a phenomenon known as countervailing incentives. Jullien (2000) develops a general framework for screening under type-dependent participation constraints, where this feature can force bunching and two-way binding of incentive constraints. Our setting differs in that outside options are endogenous, determined by competition across platforms. We show that under Common Ranking and Supermodularity, only downward incentive constraints bind (Corollary 1), so we sidestep the complications that arise in Jullien's framework and work with a standard first-order approach throughout. The distinctive feature of our analysis is not that incentive constraints bind in unusual ways, but that the anchor of the equilibrium utility profile—the type at which the profile meets the full-information benchmark—is itself determined by competition, giving rise to novel matching patterns.

Our paper also connects to the two-sided markets literature initiated by Rochet and Tirole (2003), Rochet and Tirole (2006) and Armstrong (2006). While that literature focuses on network effects and platform pricing with observable characteristics, we analyze how platforms facilitate matching when characteristics are unobservable. Weyl (2010) studies monopoly platform design with heterogeneous users, but maintains the assumption that users care only about the number, not the types, of users on the other side.

Our distinction between increasing and decreasing willingness-to-pay connects to a broad literature on private information and screening. In insurance markets, Rothschild and Stiglitz (1976) show that high-risk individuals have higher willingness-to-pay for any given coverage level, leading to positive correlation between coverage and premiums. Einav and Finkelstein (2011) survey evidence of "advantageous selection," where low-risk types have higher willingness-to-pay for any given coverage level, reversing standard predictions. We show how both cases arise naturally depending on how own type affects the payoff from matching.

A separate strand of the adverse-selection literature asks when pooling equilibria with cross-subsidization can be sustained in competitive markets. Miyazaki (1977) and Wilson (1977) developed equilibrium concepts that sustain pooling. Eisfeldt (2004) and Kurlat (2013) obtain pooling when contractually available instruments are restricted, and Daley and Green (2012) and Chang (2018) obtain pooling through search frictions. Our contribution is orthogonal to this strand: under Common Ranking and Supermodularity, our equilibrium is fully separating, with no platform cross-subsidizing across types (Proposition 1). Yet anchor matching delivers information rents to low types through the matching pattern itself.

Roadmap

Section 2 presents the model, introducing platforms that facilitate matching between privately informed agents. Section 3 provides motivating examples including marriage markets, labor markets, expertise markets, and disease transmission. Section 4 establishes our results on separation and the nature of binding incentive constraints. In Sections 5 and 6, we study the increasing and decreasing willingness-to-pay cases in detail when intermediation is free and the economic environment is symmetric. We introduce our findings on anchor matching in these sections. Section 7 develops two extensions, to an environment where the two sides of the market are asymmetric, and to one where intermediation is costly for platforms. Section 8 concludes. An analysis of the observable-type benchmark and all proofs are in the Appendix. An online supplement, available on the authors' websites, provides a general existence theorem and develops approaches to characterizing equilibrium in additional circumstances.

2 Model

2.1 Platforms and Agents

We consider a static model of a two-sided market with three sets of risk-neutral participants: a-side agents, b-side agents, and platforms. We let $s \in \{a, b\}$ denote one side of the market and $\bar{s}$ denote the other side, so if $s = a$, $\bar{s} = b$ and vice versa.

There exists a fixed measure I^s of s-side agents. Each agent has private information about their type $i \in \mathbb{I}^s$, a compact set. The distribution of types on side s is governed by an exogenous cumulative distribution function $F^s : \mathbb{I}^s \to [0, 1]$.

When a type-i, side-s agent matches with a type-j, side-$\bar{s}$ agent, their payoff is $u^s(i, j)$ before paying any platform fees. When an agent fails to match, their payoff is normalized to zero. We assume u^s is continuous and that it is continuously differentiable in its first argument at every accumulation point of $\mathbb{I}^s$. We also impose the following monotonicity assumption:

Assumption 1 (Common Ranking) *Take any $s \in \{a, b\}$, $i \in \mathbb{I}^s$, and $j, j' \in \mathbb{I}^{\bar{s}}$. If $j > j'$, $u^s(i, j) \geq u^s(i, j')$.*

The key restriction is that all agents on a given side of the market agree on the ranking of potential partners from the other side. Given this commonality in preferences, we adopt the convention of labeling types such that a higher index indicates a more desirable partner.

There is also a population of homogeneous platforms, each of which can supply intermediation effort at constant unit cost $c \geq 0$. This intermediation effort facilitates matching between agents on the two sides of the market, with costs covered through endogenous fees paid by the agents, and the equilibrium intermediation effort determined by free entry. In our leading case, intermediation is free, $c = 0$, but we also consider the possibility that intermediation effort is costly, $c > 0$.

2.2 Terms-of-trade and Payoffs

A terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b}$ specifies four objects: the fees ϕ^a and ϕ^b the platform charges to matched agents on each side and the promised distributions G^a and G^b of agent types participating on each side. We allow either ϕ^a or ϕ^b to be negative but impose that the sum of fees is nonnegative. We show below how the promised distributions G^a and G^b coordinate agents' expectations in the manner that platforms desire. Because types are private information, the terms-of-trade cannot depend on the agent's type.[5] We denote by $\mathbb{T}$ the set of feasible terms-of-trade, consisting of all tuples in which ϕ^a and ϕ^b are real numbers with (i) nonnegative sum and (ii) individual fees bounded above by the expected payoff from matching: $\phi^s \leq \int_{\mathbb{I}^{\bar{s}}} u^s(i,j)\, dG^{\bar{s}}(j)$ for all $i \in \text{supp}(G^s)$ and $s \in \{a, b\}$;[6] and G^a and G^b are probability distributions with support on subsets of $\mathbb{I}^a$ and $\mathbb{I}^b$, respectively.

Agents direct their search toward particular terms-of-trade $\tau \in \mathbb{T}$. If a type-i, side-s agent succeeds in trading at terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b} \in \mathbb{T}$ with probability $\lambda^s \in [0, 1]$, their expected payoff is

$$\bar{U}^s(i, \tau, \lambda^s) \equiv \lambda^s \left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j) dG^{\bar{s}}(j) - \phi^s \right). \tag{1}$$

That is, with probability λ^s the agent trades, receiving an expected gross payoff equal to the expected utility from a partner drawn from $G^{\bar{s}}$, minus the platform fee ϕ^s.

Platforms facilitate trades. For a platform choosing terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b} \in \mathbb{T}$ with associated matching probabilities $\lambda = (\lambda^a, \lambda^b)$, the expected gross profit per unit of intermediation effort is

$$V(\tau, \lambda) \equiv m(\lambda^a, \lambda^b)(\phi^a + \phi^b) \tag{2}$$

where $m(\lambda^a, \lambda^b)$ denotes the platform's matching probability as a function of the agents'

[5] In Appendix A, we develop a version of the model with observable types, where the terms-of-trade are allowed to depend on the agent's type.

[6] The requirement that the sum of fees is nonnegative holds for any profit-maximizing terms-of-trade when the platform matching probability is strictly positive. Likewise, the requirement that individual fees cannot exceed expected payoffs, which we call agent rationality, holds for any terms-of-trade that the agent finds optimal whenever the agent's matching probability is strictly positive.

matching probabilities, discussed in the next subsection. Each unit of intermediation effort at terms-of-trade τ results in matches with probability $m(\lambda^a, \lambda^b)$, generating revenue $\phi^a + \phi^b$. The platform's net profit subtracts the intermediation cost c from this gross profit.

2.3 Matching Function and Platform Matching Probability

To understand the platform matching probability m, we start with a more familiar object, the matching function M. We assume that in each market, the measure of matches is a constant-returns-to-scale function of the platform's intermediation effort, the measure of a-side agents, and the measure of b-side agents. Let α denote the intermediation effort and $n^s \geq 0$ denote the measure of s-side agents per unit of intermediation effort at some terms-of-trade τ. Then the measure of matches is $\alpha M(n^a, n^b)$, where $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ is nondecreasing, strictly concave, and continuously differentiable, with $M(n^a, n^b) \leq \min\{n^a, n^b\}$.

Agents' matching probability is

$$\lambda^s = L^s(n^a, n^b) \equiv \frac{M(n^a, n^b)}{n^s} \leq 1 \tag{3}$$

for $s \in \{a, b\}$ and $n^s > 0$, with boundary cases handled by taking limits. The platform matching probability m satisfies

$$m(L^a(n^a, n^b), L^b(n^a, n^b)) \equiv M(n^a, n^b),$$

and this is the m that appears in equation (2). The properties of M imply m is continuous and strictly decreasing in each argument.

By varying n^a and n^b, we trace out the feasible set of agent matching probabilities:

$$\mathbb{\Lambda} \equiv \{(L^a(n^a, n^b), L^b(n^a, n^b)) | n^a, n^b \geq 0\} \subset [0, 1]^2,$$

and let $\mathbb{\Lambda}^o \equiv \{(\lambda^a, \lambda^b) \in \mathbb{\Lambda} \mid m(\lambda^a, \lambda^b) > 0\}$. The properties of M imply that $\mathbb{\Lambda}$ is a down set (if $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ and $(\hat{\lambda}^a, \hat{\lambda}^b) \leq (\lambda^a, \lambda^b)$, then $(\hat{\lambda}^a, \hat{\lambda}^b) \in \mathbb{\Lambda}$) and that $\mathbb{\Lambda}$ is the closure of $\mathbb{\Lambda}^o$. For $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}^o$, we recover $n^s = m(\lambda^a, \lambda^b)/\lambda^s$.

Parametric Example. A CES example illustrates the setup. Suppose

$$M(n^a, n^b) = (1 + (n^a)^{-\gamma} + (n^b)^{-\gamma})^{-\frac{1}{\gamma}}, \tag{4}$$

where $\gamma > 0$ and $1/(1+\gamma)$ is the elasticity of substitution in matching between n^a and n^b. Inverting $\lambda^s = M/n^s$ for $s \in \{a, b\}$ gives

$$n^s = \left(\frac{1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma}{(\lambda^s)^\gamma} \right)^{\frac{1}{\gamma}}$$

for $s \in \{a, b\}$, and substituting back yields

$$m(\lambda^a, \lambda^b) = (1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma)^{\frac{1}{\gamma}}. \tag{5}$$

The feasible domain is $\mathbb{\Lambda} = \{(\lambda^a, \lambda^b) \geq 0 : (\lambda^a)^\gamma + (\lambda^b)^\gamma \leq 1\}$, a down set with $m = 0$ on the frontier $(\lambda^a)^\gamma + (\lambda^b)^\gamma = 1$.[7]

2.4 Equilibrium

Our equilibrium concept builds on the competitive search literature pioneered by Moen (1997) and extended to environments with private information by Guerrieri, Shimer and Wright (2010). In these models, market-makers post terms-of-trade and agents on both sides of the market direct their search toward their preferred terms. The resulting equilibrium combines price posting with rational expectations about matching probabilities.

The competitive search approach allows market participants to use posted terms-of-trade to solve selection and incentive problems. As in Moen (1997), platforms in our setting act as market-makers who compete to attract agents. However, our setting differs in two important ways. First, following Guerrieri, Shimer and Wright (2010), we allow for private information about agent types. Second, platforms must simultaneously attract agents from both sides of the market, making this a two-sided matching problem with private information.

We divide the definition of equilibrium into two parts. First, we define a partial equilibrium where everyone takes as given the agents' *equilibrium utility* $U^s(i)$, the maximum utility that a type-i, side-s agent can obtain at some terms-of-trade a platform offers. The partial equilibrium concept requires that platforms cannot profitably deviate by offering a different terms-of-trade, taking these equilibrium utilities as given. In the second step, we impose conditions that endogenously determine equilibrium utility, the maximum that an agent can attain in any active terms-of-trade.

We begin with the partial equilibrium definition:

[7] Frontier points arise as limits: $\lambda^s = 0$ from $n^s \to \infty$ with $n^{\bar{s}}$ fixed, the origin from $n^a, n^b \to \infty$ jointly, and points with $(\lambda^a)^\gamma + (\lambda^b)^\gamma = 1$ from $n^a, n^b \to 0$ at a fixed ratio.

Definition 1 *A **partial equilibrium** $\{T^p, T, \Lambda, U\}$ is two sets $T \subseteq T^p \subseteq \mathbb{T}$, a function $\Lambda = (\Lambda^a, \Lambda^b) : T^p \to \mathbb{\Lambda}$, and a pair $U = (U^a, U^b)$ with $U^s : \mathbb{I}^s \to \mathbb{R}_+$ for $s \in \{a, b\}$, such that:*

1. *(Optimal Search) for all $\tau = (\phi^s, G^s)_{s=a,b} \in \mathbb{T}$, if there exists $\lambda = (\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ satisfying*

   (a) $U^s(i) \geq \bar{U}^s(i, \tau, \lambda^s)$ for all $i \in \mathbb{I}^s$, $s \in \{a, b\}$ and

   (b) $\int_{\mathbb{I}^s} U^s(i) dG^s(i) = \int_{\mathbb{I}^s} \bar{U}^s(i, \tau, \lambda^s) dG^s(i)$ for all $s \in \{a, b\}$,

   then $\tau \in T^p$ and $\Lambda(\tau)$ satisfies (a)–(b); otherwise $\tau \notin T^p$.

2. *(Profit Maximization) $T = \arg\max_{\tau \in T^p} \bar{V}(\tau)$, where $\bar{V}(\tau) \equiv V(\tau, \Lambda(\tau))$.*

The definition has two conditions. Profit Maximization simply requires that *active* terms-of-trade $\tau \in T$ maximize profit per unit of intermediation, given the matching probabilities $\Lambda^s(\tau)$. The substantive content is in Optimal Search, which we now discuss.

Condition 1(a) requires that equilibrium utility weakly exceeds the utility that any type of agent gets at this terms-of-trade. This requirement makes sense because, were this inequality violated for some side-s, type-i agent, they would flock to this terms-of-trade. This influx would drive down these agents' matching probability until the inequality is restored. As in standard competitive search models, entry continues until agents are indifferent between this terms-of-trade and their equilibrium utility.

Condition 1(b), together with the inequality in condition 1(a), implies that agents who are supposed to come to the terms-of-trade (those in the support of G^s) obtain exactly their equilibrium utility. If expected utility were strictly less than equilibrium utility for some types in the support of G^s, the platform could not attract these agents and so could not deliver the promised type distribution G^s.

Together, conditions 1(a) and 1(b) capture the idea that platforms can coordinate expectations on who will come to the terms-of-trade, but cannot force agents to show up: G^s is the platform's announcement, and Optimal Search imposes that this announcement is self-fulfilling.[8] Terms-of-trade where no feasible matching probabilities can make the announcement self-fulfilling are excluded from T^p.

The matching probabilities $\Lambda^s(\tau)$ play a role analogous to market tightness in traditional competitive search models, but now may vary separately on both sides of the market.

[8] An alternative interpretation is that the platform posts only fees (ϕ^a, ϕ^b) and agents with equilibrium utility U direct their search across platforms. A profile $(G^a, G^b, \lambda^a, \lambda^b)$ is a Nash equilibrium of the resulting participation game at fees (ϕ^a, ϕ^b) if and only if it satisfies conditions 1(a)–(b). This game may admit multiple Nash equilibria. For example, high types may want to show up on side a if high types go to side b and vice versa, while a separate equilibrium has low types showing up on both sides. If we let the platform select its preferred Nash equilibrium of this alternative environment, the resulting definition of equilibrium is equivalent to ours.

A partial equilibrium determines which terms-of-trade could emerge given fixed equilibrium utility U. A competitive search equilibrium (CSE) endogenously determines equilibrium utility:

Definition 2 *A* ***competitive search equilibrium*** *is a partial equilibrium* $\{T^p, T, \Lambda, U\}$ *together with Radon measures* μ, ν^a, *and* ν^b *on* T *such that:*

1. *(Free Entry)* $\bar{V}(\tau) \leq c$ *for all* $\tau \in T^p$, *with equality for all* $\tau \in T$;
2. *(Consistency)* $\mu(T') = \int_{T'} \Lambda^s(\tau)\, d\nu^s(\tau)$ *for* $s \in \{a, b\}$ *and every measurable* $T' \subseteq T$;
3. *(Market Clearing)* $\forall s \in \{a, b\}$ *and* $\mathbb{I}' \subseteq \mathbb{I}^s$,

$$I^s \int_{\mathbb{I}'} dF^s(i) \geq \int_T \int_{\mathbb{I}'} dG^s(i)\, d\nu^s(\tau)$$

   with equality if $U^s(i) > 0$ *for all* $i \in \mathbb{I}'$;

4. *(Equilibrium Utility)* $\forall s \in \{a, b\}$ *and* $i \in \mathbb{I}^s$, $U^s(i) = \max\left\{0, \max_{\tau \in T} \bar{U}^s(i, \tau, \Lambda^s(\tau))\right\}$.

For measurable $T' \subseteq T$, $\mu(T')$ is the total measure of matches occurring at terms-of-trade in T', and $\nu^s(T')$ is the total measure of side-s agents matching through terms-of-trade in T'.

Free Entry ensures that platforms earn zero profit net of the intermediation cost c. If profits were positive at some $\tau \in T$, additional intermediation effort would flow to τ until profits were competed away; if negative, effort would fall.

Consistency ($d\mu = \Lambda^s\, d\nu^s$) reflects that each side-s agent at τ matches with probability $\Lambda^s(\tau)$, where equation (3) relates the measure of matches to the measure of agents at that terms-of-trade.

Market Clearing equates supply and demand for each measurable set of types, integrated against the side-s agent measure ν^s. The weak inequality allows some types to earn zero equilibrium utility and not participate.

Equilibrium Utility requires that every individual type with positive equilibrium utility actually achieves that utility at some $\tau \in T$. This condition is standard in the competitive search literature; for example, Guerrieri, Shimer and Wright (2010) introduce it as part of their optimal search condition.

2.5 Separation and Assortative Matching

We focus on two distinct characteristics of matching patterns that can emerge in equilibrium. First, we ask whether an agent participating in a particular terms-of-trade faces uncertainty about the payoff they will get from matching:

Definition 3 (Separation) *A terms-of-trade* $\tau = (\phi^s, G^s)_{s=a,b} \in \mathbb{T}$ *is separating if for* $s \in \{a, b\}$, G^s*-a.e.* k^s *and* $G^{\bar{s}}$*-a.e.* $k^{\bar{s}}, k^{\bar{s}\prime}$, $u^s(k^s, k^{\bar{s}}) = u^s(k^s, k^{\bar{s}\prime})$. *A CSE with matching measure* μ *is separating if* μ*-a.e.* $\tau \in T$ *is separating.*

In a separating CSE, an agent at any active terms-of-trade faces no uncertainty about their payoff from matching. Typically this is because each terms-of-trade attracts only one type from each side, so the agent knows exactly whom they will match with. Our definition also accommodates outcomes where multiple types participate at the same terms-of-trade, but their partners are indifferent between them.

Second, we examine whether there is systematic matching between agent types across terms-of-trade. The classical notion of assortative matching describes whether high types match with other high types:

Definition 4 (Positive Assortative Matching) *Take any subset of terms-of-trade* $T' \subset \mathbb{T}$. *Select two elements* $(\phi^s_k, G^s_k)_{s=a,b} \in T'$ *for* $k \in \{1, 2\}$ *and numbers* i_k *in the support of* G^a_k *and* j_k *in the support of* G^b_k. *If all such numbers satisfy* $(i_1 - i_2)(j_1 - j_2) \geq 0$, *then* T' *has positive assortative matching (PAM). A CSE has PAM if* T *has PAM.*

To understand this definition, consider two terms-of-trade and pick any type i_1 that participates on the a-side of terms-of-trade 1 and any type i_2 that participates on the a-side of terms-of-trade 2. Similarly, pick types j_1 and j_2 from the b-sides of these terms-of-trade. PAM requires that if $i_1 > i_2$, then $j_1 \geq j_2$. That is, if we find a higher a-side type in terms-of-trade 1 than in terms-of-trade 2, we must also find a weakly higher b-side type there.[9]

We stress that separation and assortative matching are logically distinct. An CSE may be separating and have PAM, but either or both characteristics may be absent.

3 Motivating Examples

Before characterizing the equilibrium outcomes, we introduce four motivating examples.

Marriage Market. Agents are looking for partners for heterosexual marriage. We interpret types to be hidden characteristics, and the sides of the market as men and women. The utility from forming a marriage depends on the type of both the individual and their

[9] We can analogously define Negative Assortative Matching (NAM) by requiring $(i_1 - i_2)(j_1 - j_2) \leq 0$. Under NAM, finding a higher a-side type at terms-of-trade 1 than at terms-of-trade 2 implies we must find a weakly lower b-side type there at 1 than 2. See the online supplement G for a characterization of NAM.

marriage partner. As a parametric example, we assume the utility function is CES:

$$u^s(i,j) = \left(\tfrac{1}{2} i^{\frac{\theta-1}{\theta}} + \tfrac{1}{2} j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}.$$

In this example, $\theta > 0$ is the substitution elasticity. This utility function satisfies the Common Ranking assumption. Additionally, it is supermodular and satisfies increasing willingness-to-pay, two additional assumptions which we introduce in Section 4.

Labor Market. We can also use the same structure to obtain a model of the labor market. Spence (1973) assumed that workers' human capital was unobservable, while their education choices were observable. Here we assume that both firm productivity and worker human capital are unobservable. More precisely, assume side-a firms hire side-b workers. Then $u^a(i,j)$ is the output produced when a firm with productivity i hires a worker with human capital j and $u^b(j,i)$ is the nonpecuniary amenity that the worker derives from the match.

Market for Expertise. Consider a market where privately-informed sellers trade with buyers who can imperfectly observe an asset's quality (Farboodi, Kondor and Kurlat, 2025). Side-a is the seller of an asset and side-b is the buyer. Quality is unknown to both, but a meeting between them generates a signal whose precision $\rho(i,j)$ depends on the seller's type i (how readily quality can be assessed) and the buyer's type j (expertise in valuation); the pair then jointly decides whether to trade. Higher types on either side sharpen the joint signal and thus improve the gains from trade, so the payoff function satisfies Common Ranking and increasing willingness-to-pay. Section 5.4 works out an explicit functional form and verifies these properties along with Supermodularity.

Communicable Disease. Individuals are looking to interact socially, and their types are their probability of being healthy. If a healthy individual interacts with a sick individual, they may become sick, incurring an expected cost κ, the product of the probability of getting sick and the cost of being sick. Additionally, all individuals, healthy or sick, derive utility 1 from an interaction. Sick people do not care whether they interact with sick or healthy people. All individuals know the probability that they are healthy, their type.

When a side-s agent who is healthy with probability i matches with a side-$\bar{s}$ individual who is healthy with probability j, the value of the interaction is $u^s(i,j) = 1 - \kappa i(1-j)$, because the probability that the individual gets sick is proportional to the product of the probability that they are healthy, i, and that their partner is sick, $1-j$. This payoff structure is similar to Philipson and Posner (1993) (for HIV/AIDS) and Farboodi, Jarosch and Shimer

(2021) (for COVID-19). Interestingly, while the payoff function is increasing in the partner's probability of being healthy and is supermodular, it is decreasing in the own probability of being healthy, since healthy people stand to lose more from interacting with sick people. Thus it satisfies decreasing willingness-to-pay, the flip side of increasing willingness-to-pay, another assumption we introduce in Section 4.

4 Characterization of Equilibrium

In this section, we first prove that finding a partial equilibrium is equivalent to solving a particular optimization problem. We then introduce a Supermodularity assumption on preferences and show that it ensures all CSE are separating. Finally, we establish conditions under which equilibrium utility is monotonic in agent types, allowing us to characterize equilibria through a system of differential equations when the type distribution is continuous.

4.1 The Platform's Problem

We start by recasting equilibrium as an optimization problem, as is standard in the competitive search literature. This provides a tractable way to find equilibria and establish their properties. More precisely, we show that any terms-of-trade offered in equilibrium must maximize platform profits subject to constraints which ensure that the promised mix of agents comes to the terms-of-trade.

Given nonnegative equilibrium utility U^s, we consider the following optimization problem:

$$\max_{\substack{(\lambda^a,\lambda^b)\in\mathbb{A},\\ \{\phi^s,G^s\}_{s=a,b}\in\mathbb{T}}} \quad m(\lambda^a,\lambda^b)(\phi^a+\phi^b) \tag{6}$$

$$\text{s.t. } \; U^s(i) \geq \lambda^s\left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j)dG^{\bar{s}}(j) - \phi^s\right) \;\; \forall i\in\mathbb{I}^s, s\in\{a,b\},$$

$$\int_{\mathbb{I}^s} U^s(i)dG^s(i) = \int_{\mathbb{I}^s}\lambda^s\left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j)dG^{\bar{s}}(j)-\phi^s\right)dG^s(i),\; s\in\{a,b\}.$$

The objective function is $V(\tau,\lambda)$ defined in equation (2), the platform's expected gross profit per unit of intermediation effort. The first constraint says that the equilibrium utility of a type-i, side-s agent bounds their expected payoff above, where the expected payoff is defined in equation (1). The second constraint requires that agents who do participate (those in the support of G^s) have expected payoff equal to their equilibrium utility.

Our first result establishes the close link between a solution to this optimization problem and a partial equilibrium:

Lemma 1 *Given U^a, U^b, a partial equilibrium $\{T^p, T, \Lambda, U\}$ can be constructed as follows:*

- *T^p is the set of all $\tau = (\phi^s, G^s)_{s=a,b} \in \mathbb{T}$ for which there exist $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ satisfying the constraints of problem (6);*
- *For $\tau \in T^p$, $\Lambda^s(\tau)$ is the corresponding λ^s from the constraints of (6);*
- *T is the set of $\tau \in T^p$ such that $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ solves problem (6).*

Conversely, for any partial equilibrium $\{T^p, T, \Lambda, U\}$ and any $\tau = (\phi^s, G^s)_{s=a,b} \in T$, the tuple $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ solves problem (6).

The proof is in Appendix B.1.

Unfortunately, problem (6) remains challenging to solve. It is a mathematical program with equilibrium constraints, nonconvex because increasing $dG^s(i)$ from zero changes inequality constraints into equality constraints. In the remainder of the paper, we impose additional structure on preferences in order to obtain a cleaner characterization of CSE.

4.2 Supermodularity and Separation

We are interested in finding conditions under which the CSE is separating, so each agent knows exactly what payoff they will receive at any terms-of-trade they go to. Note that Common Ranking makes separation challenging: all agents agree which partners are more desirable.[10] Supermodularity provides the solution to the problem that Common Ranking creates.

Assumption 2 (Supermodularity) *For every $s \in \{a, b\}$, $i, i' \in \mathbb{I}^s$ and $j, j' \in \mathbb{I}^{\bar{s}}$ with $i > i'$ and $j > j'$, $u^s(i, j) + u^s(i', j') > u^s(i, j') + u^s(i', j)$.*

Supermodularity implies that higher types have a stronger preference for matching with higher partner types. This ensures that by improving the quality of agents on the a-side of the market, the platform will become relatively more attractive to higher types on the b-side, and vice versa. This encourages competing platforms to cream-skim, which ensures a separating equilibrium.

When intermediation costs c are zero, we also impose a local version of Supermodularity:

Assumption 3 (Limit Supermodularity) *For every $s \in \{a, b\}$, every accumulation point i of $\mathbb{I}^s$ from below, every $j, j' \in \mathbb{I}^{\bar{s}}$ with $j > j'$, and every sequence $i'_n \uparrow i$ in $\mathbb{I}^s$ with $u^s(i, j') > u^s(i'_n, j')$ for all n, $\liminf_{n\to\infty} \frac{u^s(i,j)-u^s(i'_n,j)}{u^s(i,j')-u^s(i'_n,j')} > 1$.*

[10] Without Common Ranking, we could have $u^s(i, j) = \kappa - (i - j)^2$ for some $\kappa > 0$, so that each type prefers partners of their own type. In this case, separation would be trivial, as no one would want to deviate to a terms-of-trade designed for a different type.

Limit Supermodularity ensures sufficient complementarity when types on one side of the market are similar. Two cases are easy to check: when the type space is finite, the assumption is vacuous; and under continuous differentiability, it is implied by $u_1^s(i,j)$ strictly increasing in j at all j with $u_1^s(i,j) \geq 0$, where u_1^s denotes the partial derivative of u^s with respect to its first argument.[11] Across our leading examples, Limit Supermodularity holds whenever Supermodularity does, though this is not always true.[12] While Supermodularity plays a central role in our analysis, we only use Limit Supermodularity when intermediation costs are zero.

To prove the equilibrium is separating, Proposition 1 below establishes that under Common Ranking and Supermodularity (and Limit Supermodularity when $c = 0$), any terms-of-trade used in a CSE solves the following optimization problem:

$$\begin{aligned} \max_{(\lambda^a,\lambda^b)\in\mathbb{A},\{\phi^s,k^s\}_{s=a,b}} \quad & m(\lambda^a,\lambda^b)(\phi^a+\phi^b) \\ \text{s.t.} \quad & U^s(i) \geq \lambda^s(u^s(i,k^{\bar{s}})-\phi^s)\ \forall i<k^s, i\in\mathbb{I}^s, s\in\{a,b\} \\ & U^s(k^s) = \lambda^s(u^s(k^s,k^{\bar{s}})-\phi^s)\ \forall s\in\{a,b\}, \\ & \phi^a+\phi^b\geq 0 \text{ and } \phi^s \leq u^s(k^s,k^{\bar{s}}), s\in\{a,b\}. \end{aligned} \tag{7}$$

Problem (7) differs from problem (6) in two important ways. First, it considers only terms-of-trade in which one type k^s participates on side s. This implies that the distributions G^sin problem (6) are degenerate at a single point k^s. Second, it includes only the *downward incentive constraint*, the first constraint in the problem, which ensures that type i does not want to come to the terms-of-trade intended for type $k^s > i$. The second constraint is the *participation constraint*, which ensures that type k^s earns their equilibrium utility at this terms-of-trade. The final constraints impose nonnegativity of total fees and agent rationality, which ensures agents earn nonnegative payoffs conditional on matching. In problem (6), these were subsumed by the restriction of terms-of-trade to the feasible set $\mathbb{T}$.

Proposition 1 *Assume Common Ranking and Supermodularity. If $c = 0$, additionally assume Limit Supermodularity.*

1. *Any CSE is separating. At μ-a.e. $\tau \in T$, $\lambda^s = \Lambda^s(\tau) > 0$ on each side, and for any choice of $k^s \in supp(G^s)$ on each side, the tuple $(\lambda^s,\phi^s,k^s)_{s=a,b}$ solves problem (7) given U^s, with maximized value equal to c.*

[11] This condition is not necessary: $u^s(i,j) = (1-(1-i)^3)(1-(1-j)^3)$ with type space $\mathbb{I}^s = [0,1]$ satisfies Common Ranking, Supermodularity, and Limit Supermodularity, but $u_1^s(1,j)$ is constant and nonnegative.

[12] The payoff function $u^s(i,j) = 2(i+j)+(1-i)^2(1-j)^2$ with type space $\mathbb{I}^s = [0,1]$ satisfies Common Ranking and Supermodularity but fails Limit Supermodularity at $i = 1$.

2. *Conversely, fix a continuous* $U = (U^a, U^b)$ *with* $U^s : \mathbb{I}^s \to \mathbb{R}_+$*, a set* $T \subseteq \mathbb{T}$*, and a measure* μ *with support* T *such that:*

   (i) every $\tau = (\phi^s, G^s)_{s=a,b} \in T$ *has* G^a *and* G^b *degenerate at some* $(k^a, k^b) \in \mathbb{I}^a \times \mathbb{I}^b$*, and* $(\lambda^s, \phi^s, k^s)_{s=a,b}$ *satisfies the constraints and delivers value* c *in problem* (7) *given* U^s*;*

   (ii) setting $\Lambda^s(\tau) = \lambda^s$ *for each* $\tau \in T$ *(with* λ^s *as in (i)), there exist Radon measures* ν^a, ν^b *on* T *such that the Consistency, Market Clearing, and Equilibrium Utility conditions of Definition 2 hold;*

   (iii) problem (7) *given* U^s *has maximized value* c*.*

   Let T^p *and* Λ *extend the above to satisfy Optimal Search given* U*, and let* $\tilde{T} = \arg\max_{\tau \in T^p} \bar{V}(\tau)$*. Then* $T \subseteq \tilde{T} \subseteq T^p$*, and* $\{T^p, \tilde{T}, \Lambda, U\}$ *with measures* μ, ν^a, ν^b *is a CSE.*

The proof is in Appendix B.2. It first establishes that equilibrium utility is continuous in any CSE, using agents' ability to mimic nearby types. It then shows that Common Ranking and Supermodularity imply that for given agent matching probabilities $\lambda \in \mathbb{\Lambda}$, an increase in the types on one side of the market raises the types that can be attracted on the other side and raises the fees that can be charged. Two lemmas in the appendix package this structure: the first constructs a fixed point of the mutual best-response mapping, delivering a separating terms-of-trade satisfying all incentive constraints; the second shows that its fees are at least those of any starting point. Both parts of Proposition 1 follow.

Part 1 characterizes active markets in any CSE: each is separating, with fees and positive matching probabilities solving the finite-dimensional problem (7). This reduces the analysis of active markets to a well-behaved optimization problem.

Part 2 provides a recipe for constructing a CSE. Conjecture an equilibrium utility U^s and a set of separating terms-of-trade with the corresponding matching pattern; check that each conjectured terms-of-trade satisfies the constraints of problem (7) and delivers value c, that there exist agent measures ν^s consistent with μ for which markets clear, and that each type attains its equilibrium utility at some active market; finally, verify that no separating terms-of-trade outside the conjectured set delivers value above c. The conjecture step is still hard, but the Proposition narrows it to degenerate matching patterns and removes the need to check upward incentive constraints. Sections 5–7 show how this makes the problem tractable in special cases.

To understand why any CSE is separating, consider a terms-of-trade where multiple types participate on at least one side.[13] Common Ranking implies that all participants would weakly prefer to match with only the highest types from the other side of the market. This suggests creating a new terms-of-trade charging higher fees to attract only these high types. Supermodularity guarantees that higher types would be willing to pay more than lower types for this improved partner distribution. While such a terms-of-trade might attract even higher types than those in the original pooling terms, potentially making it infeasible, this just creates further profit opportunities. Even higher fees could be charged to attract those higher types. Any such cream-skimming deviation is profitable, contradicting free entry, so pooling cannot survive in equilibrium.[14]

A similar logic implies that only downward incentive constraints can bind in equilibrium. More precisely, we say *all upward incentive constraints are slack* in a CSE if for any active terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b} \in T$, types $k^s \in \text{supp}(G^s)$, and $j \in \mathbb{I}^{\bar{s}}$ with $u^s(k^s, j) > u^s(k^s, k^{\bar{s}})$, $U^{\bar{s}}(j) > \bar{U}^{\bar{s}}(j, \tau, \Lambda^{\bar{s}}(\tau))$. That is, if k^s has a strictly higher payoff from matching with some type j, then j must not earn their equilibrium utility at this terms-of-trade.

Corollary 1 *Assume Common Ranking and Supermodularity. If $c = 0$, additionally assume Limit Supermodularity. In any CSE, all upward incentive constraints are slack.*

The proof is in Appendix B.2. Intuitively, a binding upward incentive constraint means a higher type is just indifferent between its own terms-of-trade and one designed for a lower type. A platform can then profitably attract that higher type at a strictly higher fee, contradicting free entry. In partial equilibrium, where equilibrium utility is exogenous, upward incentive constraints may bind; but in a CSE, competition between platforms determines equilibrium utility and upward incentive constraints are slack. Slack upward incentive constraints are especially useful for characterizing equilibria with positive assortative matching (PAM), enabling us to solve for equilibrium utility recursively starting from the lowest type.

[13]A related literature examines conditions under which pooling equilibria with cross-subsidization can be sustained in competitive markets. Pooling can survive when the instruments that would enable cream-skimming are contractually restricted (Eisfeldt, 2004; Kurlat, 2013), when search or informational frictions prevent cream-skimming deviations (Daley and Green, 2012; Chang, 2018), or under equilibrium refinements that rule out destabilizing deviations (Miyazaki, 1977; Wilson, 1977). Under Common Ranking and Supermodularity, our framework delivers full separation without any of these modifications, and the information rents identified in anchor matching (Sections 5 and 6) therefore do not depend on pooling.

[14]An example illustrates the need for Supermodularity. Consider an economy with $\mathbb{I}^s = \{0, 1\}$ and payoffs $u^s(0,0) = 1$, $u^s(0,1) = 1.5$, $u^s(1,0) = 1.1$, and $u^s(1,1) = 1.2$ for $s \in \{a, b\}$. These payoffs satisfy Common Ranking but not Supermodularity. With matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ on the unit simplex ($\lambda^a \geq 0$, $\lambda^b \geq 0$, and $\lambda^a + \lambda^b \leq 1$), equilibrium utilities $U^s(0) = 0.33$ and $U^s(1) = 0.25$, and appropriate population distributions ($F^s(0) = 0.124$ and $c \approx 0.293$), we find a CSE with pooling. Specifically, a single terms-of-trade attracts both types on both sides, with $G^s(0) = F^s(0)$, $\lambda^s \approx 0.320$, and $\phi^s \approx 0.406$ for $s \in \{a, b\}$.

4.3 Monotonicity of Equilibrium Utility

To this point, we have imposed restrictions on how $u^s(i,j)$ varies with j (it is non-decreasing under Common Ranking) and how the derivative of $u^s(i,j)$ with respect to i varies with j (it is increasing under Supermodularity). We have not yet made any assumptions about how $u^s(i,j)$ varies with i. In the remainder of the paper, we find it useful to distinguish between two cases, with the qualitative characterization of equilibrium depending in important ways on which case we are in.

First, we consider the possibility that $u^s(i,j)$ is increasing in i for all j:

Assumption 4 (Increasing Willingness-to-Pay (IWTP)) $\forall\, s \in \{a,b\}$, $i > i' \in \mathbb{I}^s$, $j \in \mathbb{I}^{\bar{s}}$, $u^s(i,j) \geq u^s(i',j)$.

We call this Increasing WTP because it implies that higher types value matching with any given partner j (relative to being unmatched) more than lower types do.[15] The opposite assumption is that $u^s(i,j)$ is decreasing in i for all j:

Assumption 5 (Decreasing Willingness-to-Pay (DWTP)) $\forall s \in \{a,b\}, i > i' \in \mathbb{I}^s, j \in \mathbb{I}^{\bar{s}}$, $u^s(i,j) \leq u^s(i',j)$.

This is Decreasing WTP because higher types value matching with any given partner j less than lower types do. We refer to these assumptions collectively as Monotone WTP. Under Monotone WTP, equilibrium utility inherits the monotonicity of underlying preferences:

Lemma 2 *Assume Common Ranking. In any CSE:*

1. *If there is IWTP, then $U^s(i)$ is weakly increasing in i.*
2. *If there is DWTP, then $U^s(i)$ is weakly decreasing in i.*

The proof is again in Appendix B.3 and follows directly from participation and (upward and downward) incentive constraints.

4.4 Continuous Type Distribution and First-order Approach

In this section, we develop tools for characterizing equilibrium when the type distribution is continuous with nonempty support $\mathbb{I}^s = [\underline{i}^s, \bar{i}^s]$.

Our first step is to establish sufficient regularity of equilibrium utility to justify a local approach to incentive constraints:

[15] Theorem 1 in online supplement F uses standard competitive equilibrium machinery to establish that a CSE exists under IWTP with positive intermediation costs and finitely many types. We also show the importance of IWTP via counterexamples.

Lemma 3 *Assume Common Ranking, Supermodularity, and Monotone Willingness-to-Pay. If $c = 0$, additionally assume Limit Supermodularity. If $\mathbb{I}^s = [\underline{i}^s, \bar{i}^s]$ for $s \in \{a, b\}$, then in any CSE with matching measure μ, equilibrium utility $U^s(i)$ is almost everywhere differentiable. For μ-a.e. $\tau = (\phi^s, G^s)_{s=a,b} \in T$ such that $(k^a, k^b) \in \operatorname{supp}(G^a \times G^b)$, if $k^s \in (\underline{i}^s, \bar{i}^s)$ and U^s is differentiable at k^s,*

$$U^{s\prime}(k^s) = \Lambda^s(\tau) u_1^s(k^s, k^{\bar{s}}). \tag{8}$$

The proof is in Appendix B.3. By Proposition 1, any CSE is separating. Under Common Ranking and Supermodularity, $u^s(k^s, \cdot)$ is strictly increasing whenever $k^s > \underline{i}^s$, so at μ-a.e. $\tau \in T$, separation forces $G^{\bar{s}}$ to be degenerate at some $k^{\bar{s}}$ whenever the support of G^s contains a type above $\underline{i}^s$. Hence any interior k^s faces a single partner $k^{\bar{s}}$, and the local upward and downward incentive constraints of side s deliver equation (8). When U^s is Lipschitz, as in the CSE considered in Sections 5–7, it holds pointwise at every interior k^s and the ordinary differential equations below are pointwise identities.

For any terms-of-trade attracting types k^a and k^b with U^s differentiable at k^s, we can use this lemma to replace the downward incentive constraints in problem (7) with the *local incentive constraint*:

$$\begin{aligned} \max_{(\lambda^a,\lambda^b)\in\mathbb{A},\{\phi^s,k^s\}_{s=a,b}} \quad & m(\lambda^a, \lambda^b)(\phi^a + \phi^b) \\ \text{s.t.} \quad & U^{s\prime}(k^s) = \lambda^s u_1^s(k^s, k^{\bar{s}}) \ \forall s \in \{a, b\} \\ & U^s(k^s) = \lambda^s(u^s(k^s, k^{\bar{s}}) - \phi^s) \ \forall s \in \{a, b\}, \\ & \phi^a + \phi^b \geq 0. \end{aligned} \tag{9}$$

We can then eliminate the fees ϕ^a, ϕ^b from this problem using the participation constraint and the contact rates λ^a, λ^b using the local incentive constraint. This gives us the gross value of choosing a terms-of-trade to attract (k^a, k^b) in terms of equilibrium utility alone:

$$\hat{V}(k^a, k^b) = m\left(\frac{U^{a\prime}(k^a)}{u_1^a(k^a, k^b)}, \frac{U^{b\prime}(k^b)}{u_1^b(k^b, k^a)}\right) \sum_{s=a,b} \left(u^s(k^s, k^{\bar{s}}) - \frac{U^s(k^s) u_1^s(k^s, k^{\bar{s}})}{U^{s\prime}(k^s)}\right). \tag{10}$$

As usual, this is the product of the platform matching probability and its profit per match. To see this, note that Lemma 3 implies $\lambda^s = U^{s\prime}(k^s)/u_1^s(k^s, k^{\bar{s}})$,[16] while the participation constraint gives the fee $\phi^s = u^s(k^s, k^{\bar{s}}) - U^s(k^s)/\lambda^s$. In order for a platform choosing this terms-of-trade to obtain this value, it must satisfy three conditions. First, the combination

[16]This assumes $u_1^s(k^s, k^{\bar{s}}) \neq 0$; we treat deviations to types where u_1^s vanishes directly in the appendix.

of matching probabilities must be feasible:

$$\left(\frac{U^{a\prime}(k^a)}{u_1^a(k^a,k^b)}, \frac{U^{b\prime}(k^b)}{u_1^b(k^b,k^a)}\right) \in \mathbb{\Lambda}.$$

Second, the sum of the fees must be nonnegative:

$$\sum_{s=a,b}\left(u^s(k^s,k^{\bar{s}}) - \frac{U^s(k^s)u_1^s(k^s,k^{\bar{s}})}{U^{s\prime}(k^s)}\right) \geq 0.$$

And finally all of the downward incentive constraints must be satisfied, not just the local incentive constraint. If any of these conditions fails, there can not be any terms-of-trade which matches k^a and k^b at this level of equilibrium utility.

While the simplicity of $\hat{V}$ aids our analysis, a few challenges remain. First, we need to verify the three conditions just mentioned: feasible matching probabilities, nonnegative fees, and downward incentive compatibility. Second, we need to determine the equilibrium utilities $U^s(i)$ themselves. And third, terms-of-trade that attract the lowest types ($k^a = \underline{i}^a$ or $k^b = \underline{i}^b$) require special treatment because the lowest type has no downward incentive constraint, so the local incentive constraint that would otherwise pin contact rates does not apply. These terms-of-trade must solve the unconstrained platform problem, which then provides boundary conditions for the equilibrium utilities of higher types. Similarly, we can only compute $\hat{V}(k^a, k^b)$ at points where U^a and U^b are differentiable, which we have so far established is true almost everywhere.

In the next two sections, we tackle these challenges in two special cases, Increasing WTP (Section 5) and Decreasing WTP (Section 6). We assume throughout these sections that intermediation costs are zero, $c = 0$. This gives us a sharp characterization of necessary and sufficient conditions for a PAM CSE, as well as a sharp characterization of when matching probabilities or matching fees are used to separate types in equilibrium. In Section 7.2, we show how our results carry over to an environment with positive intermediation costs.

5 IWTP with Zero Intermediation Cost

Throughout this section, we consider an environment satisfying Common Ranking, Supermodularity, Limit Supermodularity, and IWTP. We also assume intermediation costs are zero, $c = 0$. Finally, we focus on a symmetric environment, which simplifies our exposition:

Assumption 6 (Symmetric Environment) *An environment exhibits symmetry if it has*

1. *equal populations on each side:* $I^a = I^b = I$*;*

2. *common type distributions:* $F^a = F^b = F$ *with support* $\mathbb{I}^a = \mathbb{I}^b = \mathbb{I}$*;*

3. *symmetric match utilities:* $u^a(i,j) = u^b(i,j) = u(i,j)$ *for all* i, j*; and*

4. *symmetric matching technology:* $m(\lambda^a, \lambda^b) = m(\lambda^b, \lambda^a)$*.*

We then look for a symmetric CSE with PAM, meaning it satisfies three properties:

Definition 5 (Symmetric PAM CSE) *A symmetric PAM CSE is a CSE satisfying:*

1. *type-*i*, side-*a *agents match with type-*i*, side-*b *agents for all* $i \in \mathbb{I}$*;*

2. *same-type agents on different sides have the same equilibrium utility:* $U^a(i) = U^b(i) = U(i)$ *for all* $i \in \mathbb{I}$*;*

3. *terms-of-trade for same-type agents have symmetric terms: In the active terms-of-trade* τ *that attracts type-*i *agents on both sides,* $\Lambda^a(\tau) = \Lambda^b(\tau) \equiv \ell(i)$ *and fees* $\phi^a(\tau) = \phi^b(\tau) \equiv \Phi(i)$*.*

We relax the symmetry and zero intermediation cost assumptions in Sections 7.1 and 7.2, respectively.

5.1 Platform Matching Probability

We start by showing that under IWTP and zero intermediation costs, platforms' matching probability is zero in almost every active terms-of-trade.

Lemma 4 *Assume Common Ranking, Supermodularity, Limit Supermodularity, and IWTP, as well as* $c = 0$*. In any CSE with matching measure* μ*,* $m(\Lambda^a(\tau), \Lambda^b(\tau)) = 0$ *for* μ*-a.e.* $\tau \in T$*.*

The proof is in Appendix C.1. Here we describe its logic.

To start, note that for any active terms-of-trade τ, the Free Entry condition with zero intermediation cost implies $\bar{V}(\tau) = 0$. Next, suppose there is an active terms-of-trade τ with $m(\Lambda^a(\tau), \Lambda^b(\tau)) > 0$. We prove that we can construct a nearby terms-of-trade τ' with slightly higher fees that earns positive profits. The increase in fees may attract a different type of agent, but under Common Ranking, Supermodularity, and IWTP higher types are willing to pay more to match, which implies that it can only attract a higher type. To ensure the participation constraint of the attracted type, the higher fees may need to be accompanied by a higher agent matching probability, $\Lambda^s(\tau') > \Lambda^s(\tau)$. Crucially, a small movement in this direction keeping $m(\Lambda^a(\tau'), \Lambda^b(\tau')) > 0$ is feasible, since the platform matching probability is positive under τ. Thus τ' has higher fees and a positive matching probability, implying $\bar{V}(\tau') > 0$, contradicting Free Entry.

5.2 Leading Example

To understand our general results, we turn to a parametric example. Assume CES utility and matching functions:

$$u^s(i,j) = \left(\tfrac{1}{2} i^{\frac{\theta-1}{\theta}} + \tfrac{1}{2} j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}, \qquad m(\lambda^a,\lambda^b) = (1-(\lambda^a)^\gamma - (\lambda^b)^\gamma)^{\frac{1}{\gamma}}, \tag{11}$$

where $\theta > 0$ governs the elasticity of substitution in production and $1/(1+\gamma) \in (0,1)$ is the elasticity of substitution in matching. (If $\theta = 1$, $u^s(i,j) = \sqrt{ij}$.) We introduced this payoff function in the marriage and labor market examples in Section 3. Define the maximum feasible symmetric matching probability as $\bar{\lambda} \equiv 2^{-1/\gamma}$, satisfying $m(\bar{\lambda},\bar{\lambda}) = 0$. Types are distributed uniformly on a compact interval $[\underline{i},\bar{i}]$ where $0 \le \underline{i} < \bar{i}$.

Proposition 2 below implies there is a symmetric PAM CSE if $\theta < 1+\gamma$ and $\underline{i} = 0$. The first condition parallels the observable-type benchmark (Eeckhout and Kircher, 2010): PAM requires sufficient complementarity in either production or matching. The second condition is novel to private information. When $\underline{i} > 0$, we show that the equilibrium exhibits anchor matching. When $\theta > 1+\gamma$, equilibrium exhibits NAM; we analyze this case in online supplement G.

Positive Assortative Matching When $\theta < 1+\gamma$ and $\underline{i} = 0$, there exists a symmetric CSE with the same matching pattern as under observable types—all active terms-of-trade attract the same type to both sides—but with positive fees.

By the definition of a symmetric PAM CSE, type i matches with type i in every active market. Lemma 4 tells us that the platform matching probability is driven to zero, $m(\lambda^a,\lambda^b) = 0$, and symmetry gives $\lambda^a = \lambda^b = \ell(i) = \bar{\lambda}$ in every active market. The local incentive constraint (Lemma 3) then pins down $U'(i) = \bar{\lambda} u_1(i,i) = \frac{1}{2}\bar{\lambda}$, with $U(0) = \bar{\lambda} u(0,0) = 0$ from the unconstrained problem at the lowest type. Thus $U(i) = \frac{1}{2}\bar{\lambda} i$, exactly half of the observable-type benchmark $\bar{\lambda} i$. The fee follows as a residual from the participation constraint: $\Phi(i) = u(i,i) - U(i)/\bar{\lambda} = \frac{1}{2} i$.

Though platforms earn zero profits, they need to charge fees to screen agents. Higher types pay more because, with IWTP, they value matching more and thus can be separated from lower types through fees. Free entry of platforms drives the platform matching probability to zero, dissipating profits. Additionally, a CSE requires that creating a market matching types i and j is unprofitable. The condition $\theta < 1+\gamma$ ensures that whenever the local incentive constraints hold in a match between i and $j \neq i$, the matching probabilities (λ^a,λ^b) lie outside the feasible set $\mathbb{\Lambda}$. We will see below that this same complementarity condition rules out certain deviations in the anchor case.

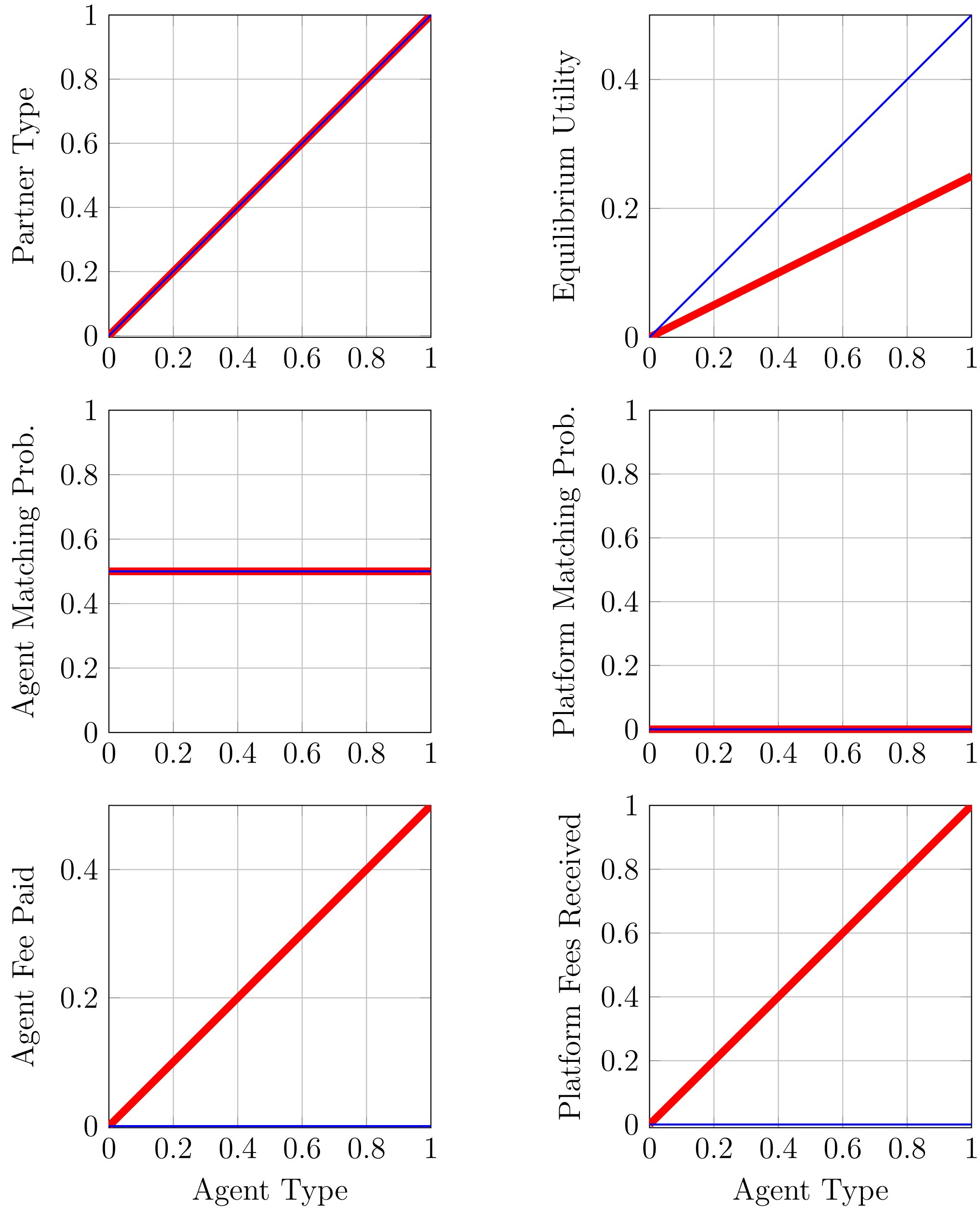

Figure 1: Positive Assortative Matching with IWTP. Notes: Thick red lines represent private-information equilibrium outcomes, while thin blue lines show observable-type equilibrium outcomes. Payoff function $u(i,j) = \sqrt{ij}$, matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$, types are distributed uniformly on $[0,1]$.

Figure 1 illustrates the CSE. The matching pattern (top left) and matching probabilities (middle row) are identical with private information and observable types (Appendix A). The key difference is in fees (bottom row): with private information, agents pay fees that consume half of match surplus; with observable types, competition drives fees to zero.

Anchor Matching When $\theta < 1+\gamma$ but $\underline{i} > 0$, PAM cannot be sustained. To understand why, suppose platforms could only create terms-of-trade that matched type i agents to type i agents. The lowest type's terms-of-trade would be undistorted, delivering $U(\underline{i}) = \bar{\lambda}u(\underline{i},\underline{i})$. For higher types, downward incentive constraints bind, yielding equilibrium utility $U(i) = \frac{1}{2}\bar{\lambda}(i+\underline{i})$ and fees $\Phi(i) = \frac{1}{2}(i-\underline{i})$.

This is inconsistent with CSE because platforms can create other terms-of-trade. The lowest type faces no downward incentive constraints, making them a uniquely valuable partner. At the candidate equilibrium utility, a platform can always profitably create a new terms-of-trade matching $\underline{i} > 0$ with some $j > \underline{i}$. This means that any CSE must have $U(\underline{i})$ higher than what a $(\underline{i},\underline{i})$ terms-of-trade could deliver. But higher $U(\underline{i})$ relaxes the incentive constraint for types just above $\underline{i}$, creating profit opportunities from matching them with strictly higher types as well. Their equilibrium utility must be higher, which relaxes constraints further up, and so on. The result is a reshuffling of partners below a threshold type $i^* \in (\underline{i},\bar{i})$. This type matches with itself and behaves exactly as an unconstrained lowest type would. Above i^*, standard PAM resumes with i^* playing the role of the lowest type.

We now describe necessary conditions for such an anchor CSE. For a given $(i,\sigma(i))$ match with $i < i^*$, local incentive constraints pin down the agent matching probabilities in a manner consistent with a zero platform matching probability (Lemma 4). Additionally, no platform can profitably create a terms-of-trade matching i with any $j \neq \sigma(i)$. For $j > \sigma(i)$, the proposed pair is more distinct than the equilibrium pairing and the local incentive constraints imply that the required agent matching probabilities lie outside the feasible set $\mathbb{\Lambda}$. This is due to the same complementarity ($\theta < 1+\gamma$) that gave us PAM when $\underline{i} = 0$. But for $j < \sigma(i)$, the matched types are more similar, and so the local incentive constraints yield contact rates with a positive platform matching probability. Instead, ruling out such a deviation requires that the sum of fees is nonpositive at any such (i,j) pairing. By continuity, the sum of fees must equal exactly zero at every active $(i,\sigma(i))$ pairing with $i \neq \sigma(i)$.

In short, if $\theta < 1+\gamma$, any active terms-of-trade where i is matched to $\sigma(i) \neq i$ must satisfy four conditions: the platform matching probability is zero, the sum of fees is zero, and the local incentive compatibility constraint is satisfied for each side of the market. These conditions form a coupled system that pins down $U(i)$, $\sigma(i)$, and $\ell^s(i)$ for $i < i^*$.[17] The

[17] The vanishing of the platform matching probability and the sum of fees does not mean that information

anchor i^* is then pinned down by the requirement that no platform can profitably create a terms-of-trade matching $\underline{i}$ with any higher type j. Equivalently, $U(\underline{i})$ is pushed up to the point where the most profitable such terms-of-trade just breaks even.

Solving this system reveals negative assortative matching in the anchor segment: $\sigma(i) > i^* > i$ for all $i < i^*$, with $\sigma(i)$ decreasing. The lowest type $\underline{i}$ requires the largest boost to utility, since its incentive constraint had the most slack, and therefore matches with the highest type in this region. As i rises towards i^*, $\sigma(i)$ declines, with $\sigma(i^*) = i^*$. The anchor i^* acts as if unconstrained. Types in the interval $(i^*, \sigma(\underline{i}))$ match both with their own type and with a lower type along the anchor segment. Equilibrium utility is higher for every type than under the conjecture at the start, with the largest gains enjoyed by the lowest type. We stress that no individual platform cross-subsidizes. Each breaks even. Rather, competition exploits the slack constraints of the lowest types, delivering them better partners and generating information rents that propagate upward.

Appendix C.2 provides the full characterization of U, σ, and the contact rates in the anchor segment, verifies downward incentive compatibility, and rules out profitable platform deviations for the parametric specification of this section. Figure 2 illustrates the anchor matching CSE when types are uniform on $[1, 2]$. Type 1 matches with type $\sigma(1) \approx 1.55$, and this boosts $U(1)$ from 0.50 (the maximum achievable in a $(1, 1)$ terms-of-trade) to above 0.53. Type 1.01 matches with 1.40, type 1.02 with 1.38, and so on, until type $i^* \approx 1.13$ matches with itself, earning equilibrium utility $\bar{\lambda} i^*$, the same as in the observable-type benchmark. Above i^*, standard PAM prevails with fees increasing in type. Types in the interval $(i^*, \sigma(1)]$ split their participation between two distinct terms-of-trade: some match assortatively with their own type, others serve as anchor partners for types below i^*.

5.3 General Conditions for PAM

This section establishes general conditions for existence of a symmetric PAM CSE. The leading example suggests that PAM requires both sufficient complementarity and a boundary condition on the lowest type; we now show this holds more generally. We leave the general analysis of anchor matching to future work.[18]

Assumption 7 *Define $\bar{\lambda}$ such that $m(\bar{\lambda}, \bar{\lambda}) = 0$.*

distortions have vanished. Local incentive constraints on both sides bind, pinning ℓ^a and ℓ^b to specific values that lie on the frontier of Λ ($m = 0$) at fees that exactly cancel ($\phi^a + \phi^b = 0$). A platform that did not face information constraints would be able to earn positive profit at this level of equilibrium utility.

[18]We conjecture that under Common Ranking, Supermodularity, Limit Supermodularity, IWTP, and Assumption 7(a), a CSE either has PAM or an anchor structure. In the latter case, there is a region pinned down by the four conditions discussed above: the platform matching probability is zero, the sum of fees is zero, and both local incentive constraints are satisfied.

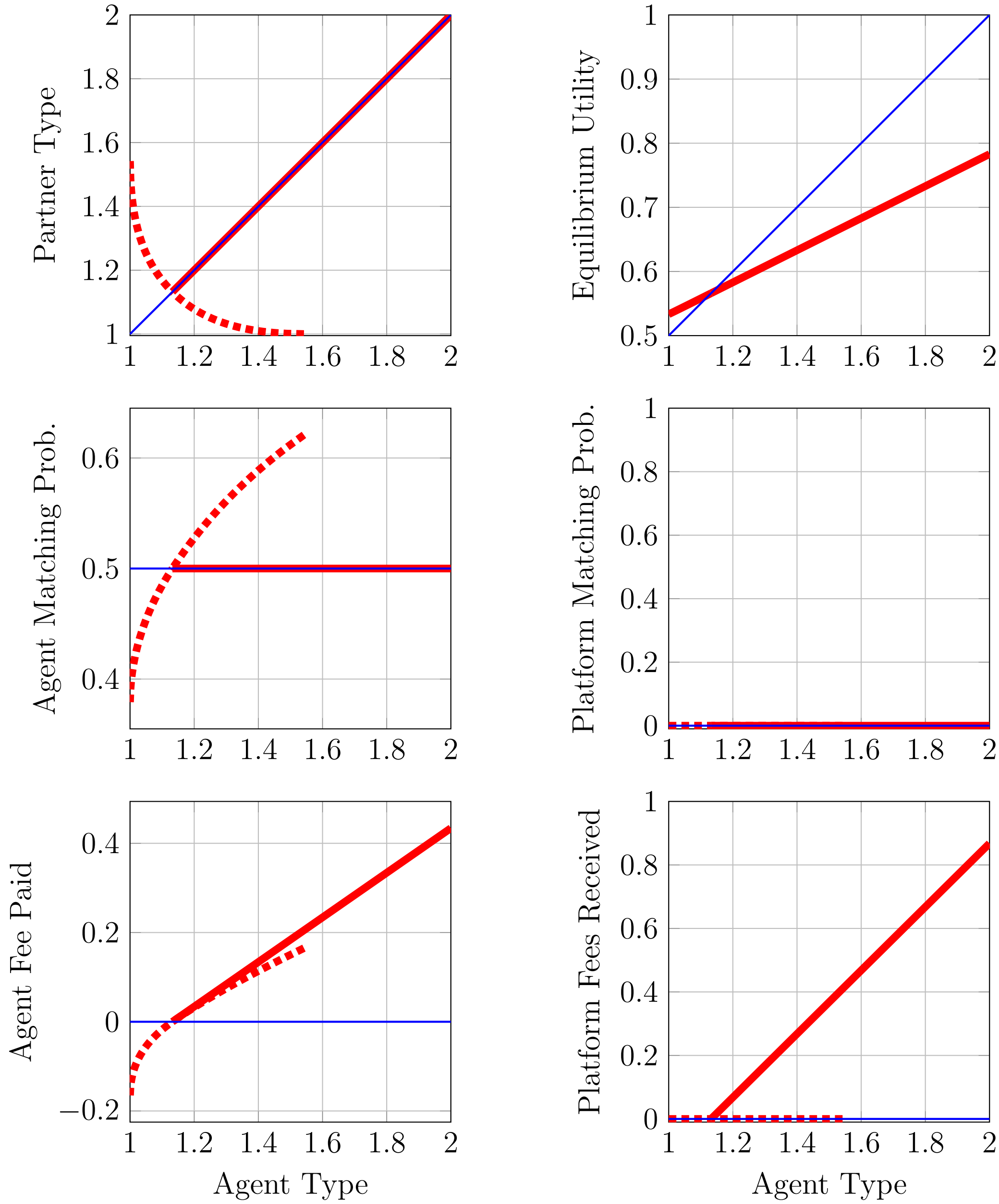

Figure 2: Anchor Matching with IWTP. Notes: Thick red solid and dotted lines represent private-information equilibrium outcomes, while thin blue lines show observable-type equilibrium outcomes. Payoff function $u(i,j) = \sqrt{ij}$, matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$, types are distributed uniformly on $[1,2]$.

(a) *For all* $i > j > \underline{i}$*,* $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}\right) \notin \mathbb{\Lambda}^o$*;*

(b) *For all* $i > \underline{i}$*, either* $u_1(i,\underline{i}) = 0$ *or* $u(\underline{i},i) + \tilde{\phi}(i) \leq 0$ *or* $\left(\tilde{\ell}(i),\, \bar{\lambda}\frac{u(\underline{i},\underline{i})}{u(\underline{i},i)+\tilde{\phi}(i)}\right) \notin \mathbb{\Lambda}^o$*, where*

$$\tilde{\phi}(i) \equiv u(i,\underline{i}) - \frac{u_1(i,\underline{i})}{u_1(i,i)}\left(u(\underline{i},\underline{i}) + \int_{\underline{i}}^{i} u_1(k,k)dk\right) \quad \textit{and} \quad \tilde{\ell}(i) \equiv \bar{\lambda}\frac{u_1(i,i)}{u_1(i,\underline{i})}.$$

Together the conditions ensure that no terms-of-trade matching a different pair (i, j) can maintain incentive compatibility while generating nonnegative fees. Assumption 7(a) rules out profitable deviations when both types are above the lower bound $\underline{i}$: the matching probabilities required by incentive compatibility push the market outside the feasible set $\mathbb{\Lambda}^o$. In our CES example, this holds if and only if $\theta < 1 + \gamma$, the same condition that gives PAM under observable types (Eeckhout and Kircher, 2010). Assumption 7(b) handles deviations involving type $\underline{i}$, who differs from other types in facing no downward incentive constraints. Here $\tilde{\ell}(i)$ is the matching probability that type i's local incentive constraint would require in an $(i, \underline{i})$ terms-of-trade, and $\tilde{\phi}(i)$ is the corresponding fee. We verify in Appendix C.1 that this additionally requires $\underline{i} = 0$ in our CES example.

We now state our general characterization of PAM with IWTP:

Proposition 2 *Assume Common Ranking, Supermodularity, Limit Supermodularity, IWTP, Symmetric Environment, and Assumption 7. If* $c = 0$*, there is a symmetric PAM CSE. For the terms-of-trade* τ *attracting type* i *agents, the matching probability is* $\Lambda(\tau) = \bar{\lambda}$*; the fee solves* $\Phi'(i) = u_2(i,i)$ *with* $\Phi(\underline{i}) = 0$*; and equilibrium utility solves* $U'(i) = \bar{\lambda}u_1(i,i)$ *with* $U(\underline{i}) = \bar{\lambda}u(\underline{i},\underline{i})$*.*

The proof is in Appendix C.1. In a symmetric PAM CSE with $c = 0$, type i matches with type i, and Lemma 4 and symmetry give $\lambda = \bar{\lambda}$ in every active market. These two facts, together with the local incentive constraint and the absence of distortions at the lowest type, pin down equilibrium utility and fees. We then verify that downward incentive constraints hold and use Assumption 7 to rule out profitable deviations to markets matching different types.

Conditional on having a PAM equilibrium, the equilibrium outcomes resemble the ones from the parametric case we just described. All types have the maximum symmetric matching probability $\bar{\lambda}$. Since higher types gain more from matching with IWTP, trying to exclude lower types by rationing matches is counterproductive. On the other hand, fees are increasing to ensure that lower types, who gain less from matching with a higher type, prefer matching with their own type at a lower fee to paying the higher fee for a higher-type partner.

5.4 Application: Market for Expertise

We close this section by working out the expertise example introduced in Section 3. Sellers hold a single asset which may be of two qualities, S or B, with equal prior probability. Quality S assets are naturally retained by the seller, while quality B assets are more valuable in the hands of the buyer. For example, the asset may be a mortgage with quality reflecting servicing difficulty: the seller (mortgage issuer) is well-placed to handle a hard-to-service loan, while the buyer (investment bank) has the deep pockets to hold other loans. If the seller ultimately holds a quality S asset, their payoff is 1, while it is -1 if they hold a quality B asset; the buyer's payoffs are reversed.

Neither party observes quality directly, but a type-i seller and type-j buyer jointly observe a normally distributed public signal: mean -1 for quality S, mean $+1$ for quality B, with precision $\rho = ij$ in both cases. The pair then jointly decide whether to trade and make any specified transfers. Trade is jointly profitable if and only if the signal is positive, so the pre-transfer expected payoff of each side from a meeting between a type-i seller and type-j buyer is $u(i,j) = \frac{1}{2} - \Psi(1, ij)$, where $\Psi(1,\rho)$ is the probability that a normally distributed random variable with mean 1 and precision ρ is negative.[19] The product form $\rho = ij$ makes observability and expertise complements in producing information.

Since $\Psi(1,\rho)$ is decreasing in ρ, the payoff is increasing in both arguments, so Common Ranking and IWTP both hold. The cross-partial $u_{12}(i,j)$ has the same sign as $1 - ij$, so we restrict the type space to $[0,1]$, where Supermodularity holds. Assumption 7(a) is satisfied only on a tighter interval $[0,\bar{i}]$: with a CES matching function with parameter γ, we require $\bar{i} \le \sqrt{\frac{1+\gamma-\sqrt{1+2\gamma}}{\gamma}} < 1$. Assumption 7(b) holds since $u_1(i,0) = 0$, so the lowest type $\underline{i} = 0$ has no incentive to deviate to a higher-type partner.

Whenever the support of the type distribution is $[0,\bar{i}]$, Proposition 2 therefore delivers a symmetric PAM CSE. Each type matches at the maximum symmetric rate $\bar{\lambda} = 2^{-1/\gamma}$, fees consume half of equilibrium output, $\Phi(i) = u(i,i)/2$, and equilibrium utility $U(i) = \bar{\lambda} u(i,i)/2$ is half the observable-type benchmark. Even though buyer and seller play asymmetric economic roles, the joint precision ij enters their payoffs symmetrically, so the screening pattern matches the marriage-market example: high types pay high fees to meet each other, while low types match cheaply with each other.

[19] If the asset is quality S, with probability $\frac{1}{2}$, trade occurs with probability $1 - \Psi(-1, ij) = \Psi(1, ij)$, resulting in payoff -1 for the buyer. If the asset is of quality B, also with probability $\frac{1}{2}$, trade occurs with probability $1 - \Psi(1, ij)$, resulting in payoff 1 for the buyer. Adding these up gives the buyer's expected payoff; the seller's is symmetric.

6 DWTP with Zero Intermediation Cost

This section parallels the previous one, but examines environments with DWTP. We assume Common Ranking and Supermodularity throughout,[20] again focusing on a Symmetric Environment with zero intermediation costs.

6.1 Sum of Fees

The DWTP analog of Lemma 4 concerns the sum of fees rather than the platform matching probability.

Lemma 5 *Assume Common Ranking, Supermodularity, and DWTP, as well as $c = 0$. In any CSE with matching measure μ, $\phi^a + \phi^b = 0$ for μ-a.e. $\tau \in T$.*

The proof is in Appendix D.1. The logic mirrors Lemma 4, with the roles of fees and matching probabilities reversed. Suppose an active terms-of-trade τ has $\phi^a(\tau) + \phi^b(\tau) > 0$. Construct a nearby terms-of-trade τ' with slightly lower fees: agents are willing to participate at a lower contact rate, which ensures the platform has a positive matching probability $m(\Lambda^a(\tau'), \Lambda^b(\tau')) > 0$. The lower fees may attract different types, but under Common Ranking, Supermodularity, and DWTP they can only attract higher types, those most willing to accept a reduction in matching probability for a reduction in fees. Since partners will pay more to match with a higher type (Common Ranking), this may permit an increase in fees. Thus τ' has positive fees and a positive matching probability, contradicting Free Entry.

6.2 Leading Example

We again illustrate our results with a parametric example. Let $u^s(i, j) = 1 - \kappa i(1 - j)$ for $s \in \{a, b\}$, with support on some interval $\mathbb{I} \subseteq [0, 1]$ and $\kappa > 0$. This is our model of disease transmission from Section 3. A type-i individual is healthy with probability i. They gain 1 from a match but suffer expected loss κ if they contract the disease from a sick partner, which happens if the individual is healthy and their partner is sick. This function satisfies Common Ranking, Supermodularity, and DWTP. We again assume a CES matching function with elasticity of substitution $1/(1 + \gamma)$. Proposition 3 below implies there is a symmetric PAM CSE if $\kappa \leq \min\{2(1 + \gamma), 4\}$ and $\underline{i} = 0$. As in the IWTP case, PAM requires both sufficient complementarity and that the lowest type be at the boundary. We also illustrate an anchor matching CSE when $\kappa = 2$, $\gamma = 1$, and $\underline{i} > 0$.

[20] Under DWTP, Limit Supermodularity is always satisfied.

Positive Assortative Matching When $\kappa \leq \min\{2(1+\gamma), 4\}$ and $\underline{i} = 0$, each type i matches with type i. Lemma 5 gives $\phi^a + \phi^b = 0$, and symmetry gives $\Phi(i) = 0$. The unconstrained $(0,0)$ market, the participation constraint $U(i) = \ell(i)u(i,i)$, and the local incentive constraint $U'(i) = \ell(i)u_1(i,i)$ pin down equilibrium utility $U(i)$ and the matching probability $\ell(i)$.

With DWTP, separation works through rationing rather than fees. Higher types value matching less, so match with lower probability; this deters lower types, who are less willing to accept the risk of not matching. Fees cannot separate types in this environment, since lower types would be willing to pay any fee that higher types would accept.

Figure 3 illustrates this equilibrium. Fees are zero in all active terms-of-trade (bottom row), in contrast to the IWTP case. Instead, matching probabilities decline with type (middle left): healthier individuals match less frequently to avoid sick partners. The platform matching probability increases with type (middle right), and platforms break even because they collect no fees.

Anchor Matching When $\kappa \leq \min\{2(1+\gamma), 4\}$ but $\underline{i} > 0$, PAM cannot be sustained. The logic parallels the IWTP case: the lowest type faces no downward incentive constraints, creating profit opportunities. A CSE requires a higher level of $U(\underline{i})$ than any $(\underline{i}, \underline{i})$ terms-of-trade could deliver, which requires matching $\underline{i}$ with higher types. This relaxes constraints for types just above $\underline{i}$, generating a cascade of increases in equilibrium utility.

As in the IWTP case, we find that if $\kappa \leq \min\{2(1+\gamma), 4\}$, any active terms-of-trade where i is matched to $\sigma(i) \neq i$ must satisfy four conditions: sum of fees is zero, platform matching probability is zero, and each of the local incentive constraints is satisfied. The first two conditions are necessary to ensure that matching different pairs is unprofitable. The local incentive compatibility constraints ensure the downward incentive constraints hold. In Appendix D.2, we show how to use these conditions to find the CSE. This analysis again reveals negative assortative matching at the bottom of the type distribution.

Figure 4 illustrates this equilibrium when types are uniform on $[0.2, 0.4]$. Type 0.2 matches with type $\sigma(0.2) \approx 0.27$, boosting $U(0.2)$ from 0.34 (the observable-type benchmark) to 0.35. The cascade continues until type $i^* \approx 0.22$, which matches with itself. Above i^*, PAM prevails, with matching probabilities declining in type. Types in $(i^*, \sigma(0.2))$ split between the two regions.

6.3 General Conditions for PAM

This section establishes general conditions for existence of a symmetric PAM CSE, paralleling the IWTP case in Section 5. The leading example suggests that PAM requires both sufficient

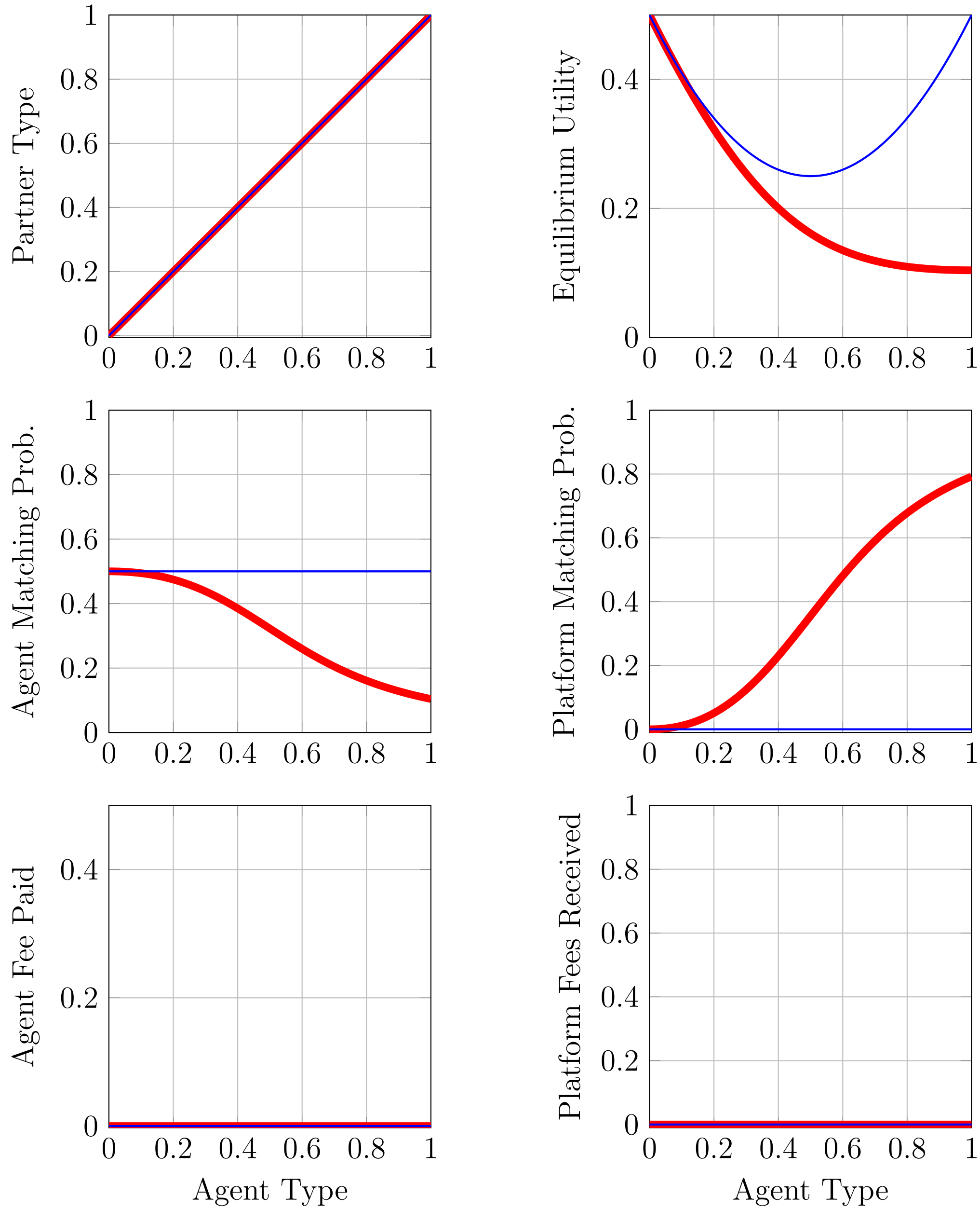

Figure 3: Positive Assortative Matching with DWTP. Notes: Thick red lines represent private-information equilibrium outcomes, while thin blue lines show observable-type equilibrium outcomes. Payoff function $u(i, j) = 1 - 2i(1 - j)$, matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$, types are distributed uniformly on $[0, 1]$.

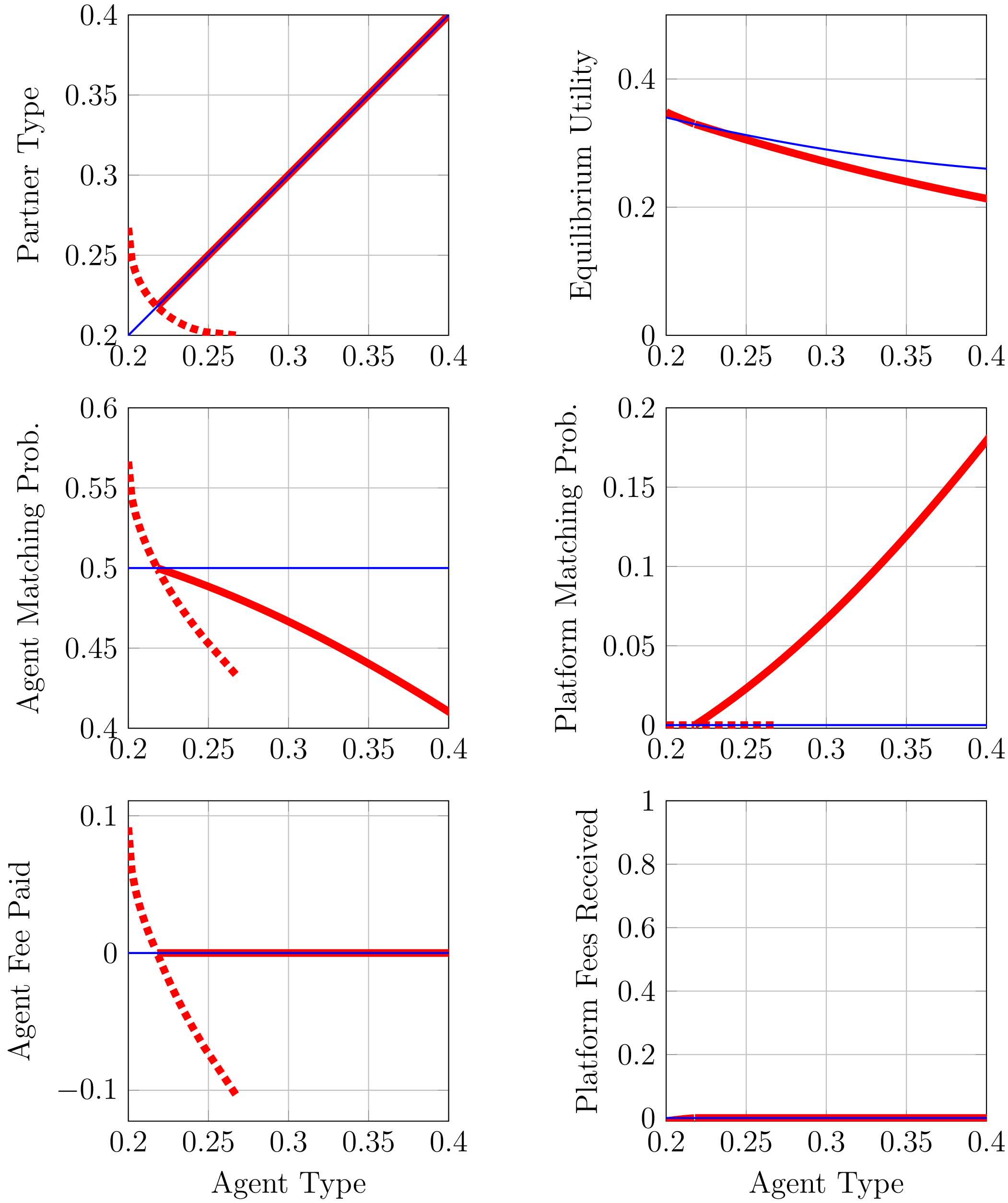

Figure 4: Anchor Matching with DWTP. Notes: Thick red solid and dotted lines represent private-information equilibrium outcomes, while thin blue lines show observable-type equilibrium outcomes. Payoff function $u(i,j) = 1 - 2i(1-j)$, matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$, types are distributed uniformly on $[0.2, 0.4]$.

complementarity and a boundary condition on the lowest type; we now show this holds more generally. We again leave the general analysis of anchor matching to future work.

Assumption 8 *Define $\bar{\lambda}$ such that $m(\bar{\lambda}, \bar{\lambda}) = 0$.*

(a) For all $i > j > \underline{i}$, $u(i,j) + u(j,i) \leq u(i,i)\frac{u_1(i,j)}{u_1(i,i)} + u(j,j)\frac{u_1(j,i)}{u_1(j,j)}$;

(b) For all $i > \underline{i}$, either $u(\underline{i}, i) + \hat{\phi}(i) \leq 0$ or $\left(\bar{\lambda}\frac{u(\underline{i},i)}{u(\underline{i},i)+\hat{\phi}(i)}, \hat{\ell}(i)\right) \notin \Lambda^o$, where

$$\hat{\phi}(i) \equiv u(i, \underline{i}) - \frac{u_1(i,\underline{i})}{u_1(i,i)}u(i,i) \quad \textit{and} \quad \hat{\ell}(i) \equiv \bar{\lambda}\frac{u(\underline{i},\underline{i})u_1(i,i)}{u(i,i)u_1(i,\underline{i})}\exp\left(\int_{\underline{i}}^{i} \frac{u_1(k,k)}{u(k,k)}dk\right).$$

Again, Assumption 8(a) ensures that creating a terms-of-trade matching i and j with $i > j > \underline{i}$ is not profitable. Assumption 8(b) rules out the profitability of a terms-of-trade matching $i > \underline{i}$ with $\underline{i}$. In our disease example, condition (a) is always satisfied, and condition (b) holds when $\kappa \leq \min\{2(1+\gamma), 4\}$ and $\underline{i} = 0$; see Appendix D.1.

Proposition 3 *Assume Common Ranking, Supermodularity, DWTP, Symmetric Environment, and Assumption 8. If $c = 0$, there is a symmetric PAM CSE. For the terms-of-trade τ attracting type i agents, the matching probability is $\ell(i)$ solving $\frac{\ell'(i)}{\ell(i)} = -\frac{u_2(i,i)}{u(i,i)}$, with $\ell(\underline{i}) = \bar{\lambda}$; the fee is $\phi^a = \phi^b = 0$; and equilibrium utility solves $\frac{U'(i)}{U(i)} = \frac{u_1(i,i)}{u(i,i)}$, with $U(\underline{i}) = \bar{\lambda}u(\underline{i}, \underline{i})$.*

The proof is in Appendix D.1 and follows the same structure as the proof of Proposition 2. The general characterization is similar to our example: fees are zero, while low matching probabilities are used to exclude lower types from terms-of-trade intended for higher types.

7 Extensions

Sections 5 and 6 focused on symmetric environments with zero intermediation costs. We now illustrate how the analysis extends along two dimensions: asymmetry between the two sides of the market (Section 7.1) and positive intermediation costs (Section 7.2). Online supplement G carries out a parallel analysis for NAM with zero costs, and online supplement H combines both extensions, characterizing equilibrium in asymmetric environments with positive costs. These examples illustrate how one can numerically construct CSE with bilateral private information: in each case, we characterize equilibrium as the solution to a system of differential equations and verify it numerically.

7.1 Asymmetric Environment

Our previous analysis assumed that the a and b sides of the market are symmetric. We now relax that assumption. In Appendix E.1, we show how to characterize an equilibrium as the solution to a system of differential equations. Here we illustrate a particular example. We assume that the payoff functions are $u^a(i,j) = i^{0.9}j^{0.1}$ and $u^b(j,i) = j^{0.1}i^{0.9}$, so the type of the side-a agent is more important for both agents' payoff. Types on each side are distributed uniformly on $[0,1]$ with equal populations. Similar to the baseline cases, we assume the platform matching probability is $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$.

The blue lines in Figure 5 illustrate the case with observable types. Asymmetric payoffs lead to asymmetric fees, matching probabilities, and partner types. For example, it is important to make sure that the highest types on side a match. Thus they have a high matching probability, which requires more agents on side b than on side a. This in turn means that all but the highest side a agents match with lower side b agents than themselves. Side b agents enjoy the higher matching partner, but then pay for it through higher fees, while many side a agents receive a subsidy, i.e. negative fees. Finally, we find that there is no profit maximizing terms-of-trade that attracts the lowest side a agents, and so they earn zero utility.

The red lines in Figure 5 illustrate the case with private information. In addition to the trade-offs from the full-information setting, equilibrium outcomes also reflect the need to screen. The starkest difference is in the fees. Because side-a types matter more for surplus, they have a bigger incentive to misrepresent their type. Screening thus requires large positive fees charged to these agents, in contrast to the subsidies they receive with observable types.

Private information also affects matching probabilities, partner types, and utility. High-type side-a agents have even higher matching probabilities, leading to more low-type side-a agents being rationed. On the b-side, everyone has a lower matching probability. Expected partner types shift in the same direction: a-types obtain slightly worse partners than in the observable-type benchmark, whereas low b-types obtain better partners. Equilibrium utility is lowered for all types on either side. Finally, platforms receive positive fees in equilibrium, as in the symmetric case, but the fees are mostly paid by high types on side a.

7.2 Positive Intermediation Costs

We now consider the case where intermediation is costly, $c > 0$. For simplicity, we assume a Symmetric Environment and look for a symmetric PAM CSE. We consider a PAM CSE in an asymmetric environment in online supplement H.1, and NAM in online supplement H.2.

We find that both rationing and fees are used in equilibrium, but whether higher types are more or less rationed and pay higher or lower fees depends on the direction of willingness-

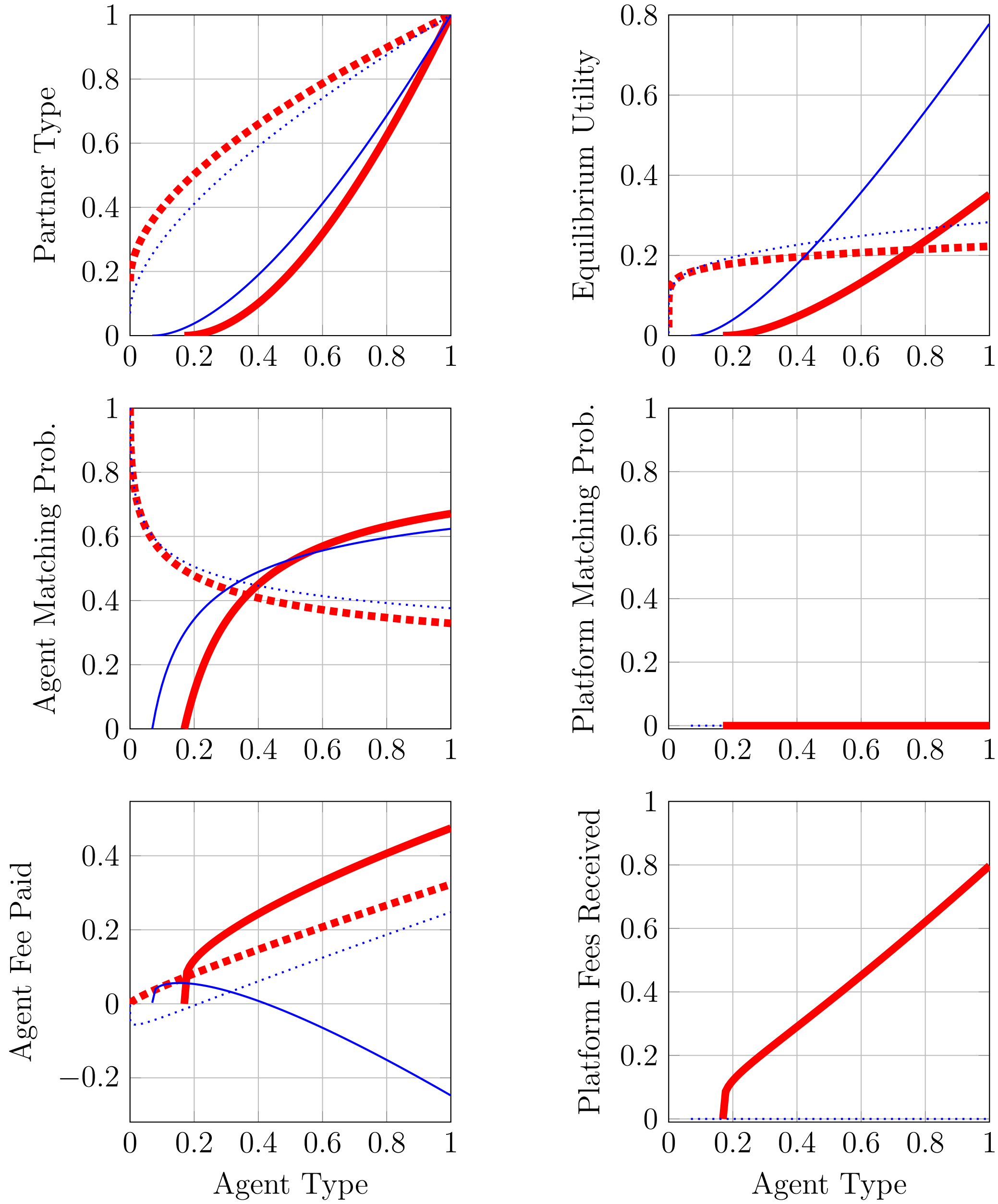

Figure 5: Positive Assortative Matching with IWTP and Asymmetric Payoff Functions. Notes: Red solid lines represent private-information equilibrium outcomes for a-side agents, red dotted lines represent private-information equilibrium outcomes for b-side agents, and blue solid and dotted lines show observable-type equilibrium outcomes for the two types. Payoff functions $u^a(i,j) = i^{0.9}j^{0.1}$, $u^b(j,i) = j^{0.1}i^{0.9}$; matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$; types are distributed uniformly on $[0,1]$.

to-pay. To see why, start from the lowest type. There is no information problem for these agents, since incentive constraints only bind downward (Corollary 1). But with $c > 0$, platforms must cover their intermediation costs, which requires both some rationing ($\lambda < \bar{\lambda}$) and positive fees ($\phi > 0$). Now consider what happens as we move up the type distribution. With IWTP, higher types face higher fees for the same reason as when $c = 0$: fees exclude lower types who value matching less. But higher fees mean more revenue per match. For platforms to still break even, their matching probability must be lower when they attract higher types, or equivalently, the higher types must have a higher matching probability. With DWTP, the logic reverses: higher types are separated through lower matching probabilities, which raises the platform matching probability. For platforms to still break even, higher types must pay lower fees.

To state this formally, first define

$$h(\lambda, i) \equiv \lambda \left(u(i,i) - \frac{c}{2m(\lambda, \lambda)} \right), \tag{12}$$

a type i agent's utility in an (i,i) market with matching probability λ and intermediation cost c. Let $\ell^*(i) \equiv \arg\max_{\lambda \in [0,\bar{\lambda}]} h(\lambda, i)$ denote the matching probability that would prevail absent incentive constraints. This leads to our characterization of a PAM CSE with positive intermediation costs. For expositional simplicity, we assume that costs are low enough that an unconstrained (i,i) market would cover them for every type,

$$u(i,i) > \frac{c}{2\,m(0,0)} \text{ for all } i. \tag{13}$$

This ensures all types participate and earn positive equilibrium utility in a PAM CSE:

Proposition 4 *Assume Common Ranking, Supermodularity, Monotone WTP, and a symmetric environment. Also assume c is positive and satisfies inequality (13). If there exists a symmetric PAM CSE, then:*

1. *The matching probabilities $\ell(i)$ solve*

$$h_1(\ell(i), i)\frac{\ell'(i)}{\ell(i)} = -u_2(i,i), \tag{14}$$

with boundary condition $\ell(\underline{i}) = \ell^*(\underline{i})$;

2. *Equilibrium utility satisfies $U(i) = h(\ell(i), i)$;*

3. *Fees satisfy $\Phi(i) = u(i,i) - U(i)/\ell(i)$;*

4. These solutions satisfy all global incentive constraints.

Moreover, with

- *IWTP, $U(i)$, $\ell(i)$, and $\Phi(i)$ are increasing in i and $\ell(i) > \ell^*(i)$ for all $i > \underline{i}$;*
- *DWTP, $U(i)$, $\ell(i)$, and $\Phi(i)$ are decreasing in i and $\ell(i) < \ell^*(i)$ for all $i > \underline{i}$.*

The proof, in Appendix E.2, shows that any symmetric PAM CSE must solve a platform optimization problem whose free-entry and local incentive conditions reduce to the ordinary differential equation (14). Several technical subtleties arise: the differential equation degenerates at $\underline{i}$ (where the problem is unconstrained), there are two solution branches (one increasing, one decreasing in ℓ), and one must verify continuity of ℓ and positivity of equilibrium objects. The proof resolves each of these and shows that IWTP selects the increasing branch while DWTP selects the decreasing one.

Proposition 4 has several notable implications. First, the differential equation characterization of equilibrium has remarkable simplicity and power. While we only imposed local incentive constraints, the resulting allocation automatically satisfies global incentive constraints. This relies on the way that CSE pins down equilibrium utility, which functions as agents' outside option.

Second, the direction of Monotone Willingness-to-Pay determines the structure of equilibrium utilities, contact rates, and fees. With IWTP, higher types receive higher utilities and face higher contact rates and fees. With DWTP, the pattern reverses. This stands in contrast to the case with observable types, where the direction of these patterns depends on whether the equilibrium payoff $u(i, i)$ is increasing or decreasing, not on the willingness-to-pay.

Third, private information systematically alters allocations relative to the observable-type benchmark. With IWTP, equilibrium contact rates exceed the full-information optimum $\ell^*(i)$. The free-entry condition then implies that fees must also be higher to cover platform costs. With DWTP, this flips: both contact rates and fees fall below their full-information levels, since platforms adjust contact rates and fees to make mimicking less attractive to lower types.

While Proposition 4 characterizes symmetric PAM equilibria when they exist, it does not guarantee existence. The candidate equilibrium utility satisfies all incentive constraints, but we must still verify that no platform would prefer to create a terms-of-trade matching different types. When $c = 0$, Propositions 2 and 3 provide sufficient conditions, but we have not derived analogous conditions when $c > 0$. Verification reduces to checking $\hat{V}(k^a, k^b)$ in equation (10), a straightforward numerical computation. Figures 6 and 7 illustrate numerically verified equilibria for the baseline IWTP and DWTP models with positive costs.

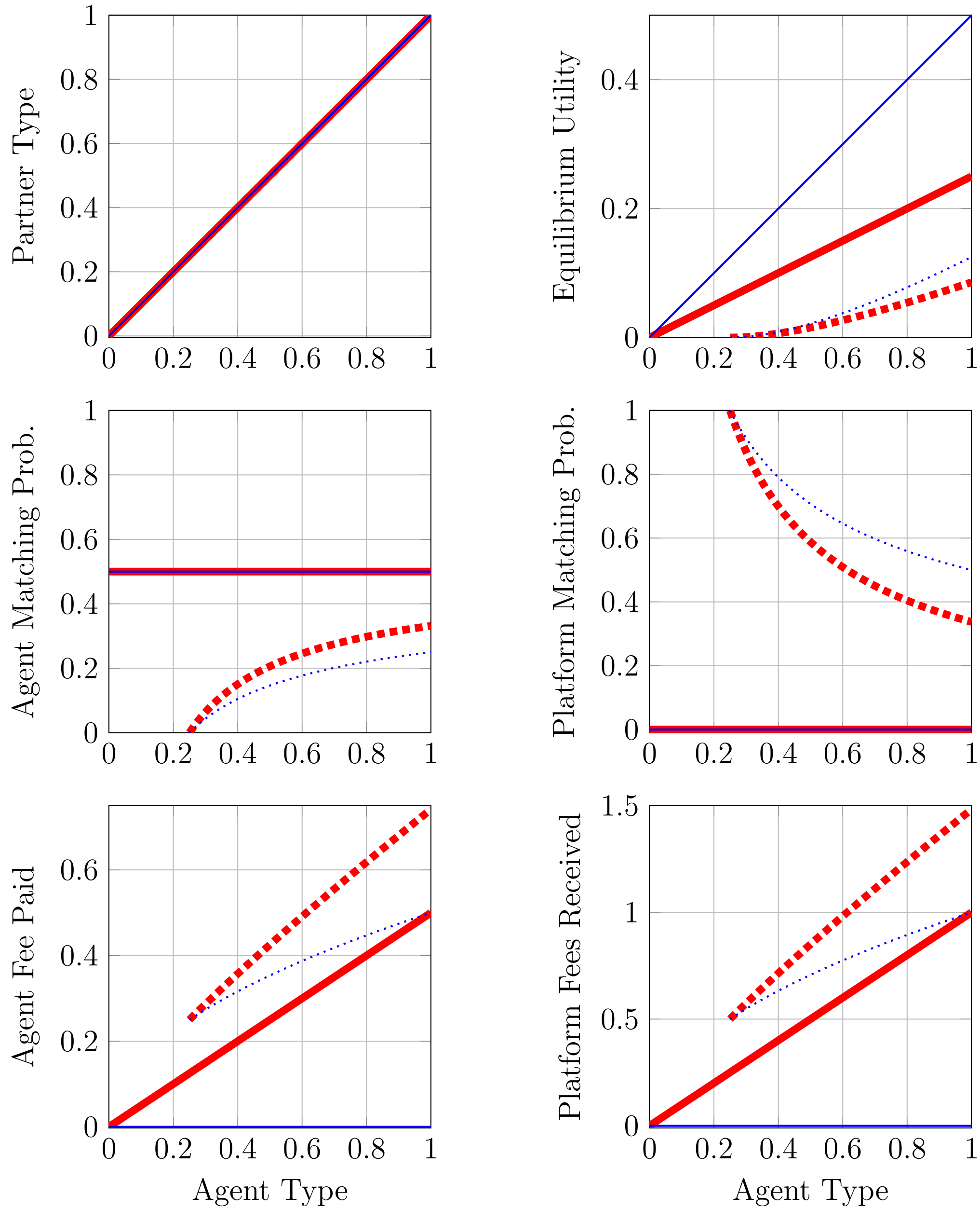

Figure 6: Positive Assortative Matching with IWTP and Positive Intermediation Costs. Notes: Thick red lines represent private-information equilibrium outcomes, thin blue lines show observable-type equilibrium outcomes. Solid lines have $c = 0$, dashed lines have $c = 0.5$. Payoff function $u(i, j) = \sqrt{ij}$, matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$, types are distributed uniformly on $[0, 1]$.

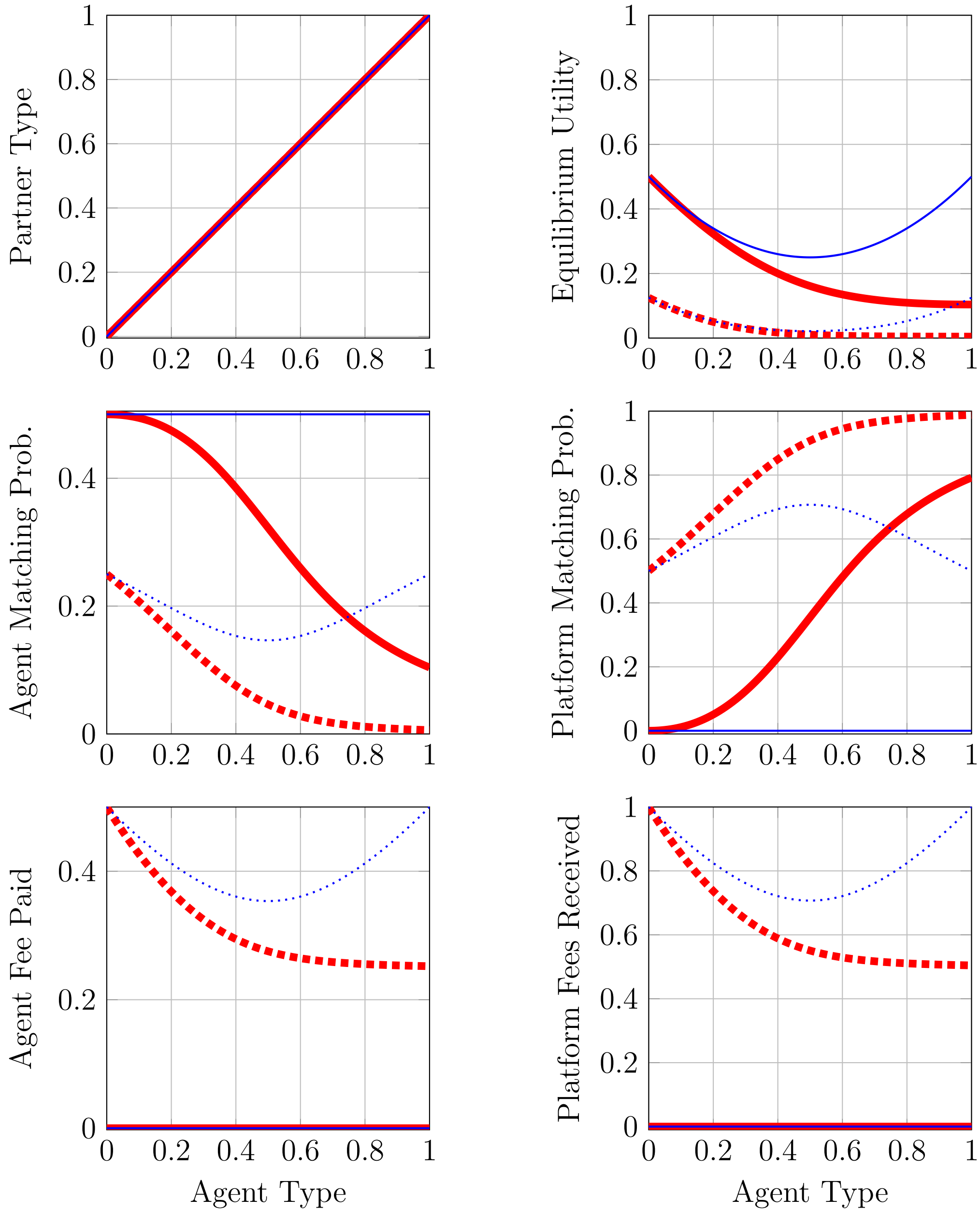

Figure 7: Positive Assortative Matching with DWTP and Positive Intermediation Costs. Notes: Thick red lines represent private-information equilibrium outcomes, thin blue lines show observable-type equilibrium outcomes. Solid lines have $c = 0$, dashed lines have $c = 1$. Payoff function $u(i, j) = 1 - 2i(1 - j)$, matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$, types are distributed uniformly on $[0, 1]$.

8 Concluding Remarks

We have developed a competitive framework for matching markets in which agents on both sides hold partner-relevant private information. The framework delivers sharp predictions under standard monotonicity and complementarity conditions: competition drives the market toward separation, only downward incentive constraints bind, and the mechanism of separation is determined by the monotonicity of agents' willingness-to-pay.

When higher types have a higher willingness-to-pay, competition separates agents through fees paid to platforms, preserving high matching rates. When higher types have a lower willingness-to-pay, the market relies on rationing to screen types. This dichotomy suggests that observed frictions, such as unemployment queues or fees paid to headhunters, may reflect equilibrium screening devices arising from the underlying information structure.

The framework also refines standard intuitions about redistribution in competitive environments. The classical view is that cream-skimming prevents cross-subsidization under adverse selection. We find that competition can nonetheless deliver information rents to low types: in anchor matching, the lowest types are paired with substantially higher-type partners, and the resulting rents propagate upward through local incentive constraints. Low types can therefore obtain utility above the full-information benchmark. The mechanism is not classical cross-subsidization within a contract; it is the endogenous determination of the anchor i^*, the type at which the equilibrium utility profile touches the full-information benchmark. When this anchor lies above the bottom of the support of the type distribution, all types below it enjoy rents.

By characterizing the terms-of-trade, this framework provides a mapping from observable market features to the underlying information structure. In the presence of bilateral private information, competitive markets organize trade not only by adjusting prices and by rationing, but by changing who matches with whom.

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Appendix

A Model with Observable Types

In this section, we define and characterize equilibrium when types are observable. The only modification to the baseline model is that terms-of-trade can condition on an agent's type. A terms-of-trade is now $\tau = (\phi^s, G^s)_{s=a,b}$, where $\phi^s : \mathbb{I}^s \to \mathbb{R}$ is a type-contingent fee schedule and G^s is the promised type distribution on side s, for $s \in \{a, b\}$. Let $\mathbb{T}^O$ denote the set of such feasible terms-of-trade. The definitions of partial and competitive search equilibrium carry over from the body with $\mathbb{T}^O$ in place of $\mathbb{T}$, so we do not repeat them here.

Given equilibrium utilities U^a, U^b, the analog of problem (6) is

$$\max_{(\phi^s,\lambda^s,G^s)_{s=a,b}} m(\lambda^a, \lambda^b) \sum_{s=a,b} \int_{\mathbb{I}^s} \phi^s(i)\, dG^s(i) \tag{15}$$
$$\text{s.t. } U^s(i) \geq \lambda^s \left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j)\, dG^{\bar{s}}(j) - \phi^s(i) \right) \ \forall i \in \mathbb{I}^s, s \in \{a, b\},$$
$$\text{with equality if } i \in \text{supp}(G^s).$$

As in the private-information case, the first constraint is the individual participation constraint and the second-line equality requires that types in the support of G^s earn exactly their equilibrium utility. The difference from (6) is that the fee $\phi^s(i)$ now varies with the agent's type.

It is without loss of generality to focus on separating terms-of-trade, in which each G^s is degenerate at a single type k^s because a non-separating terms-of-trade can be replicated by a collection of separating ones at the same matching probabilities. More formally:

Lemma 6 *Consider a CSE with observable types. Either every profit-maximizing terms-of-trade is separating, or there is an equivalent CSE in which every profit-maximizing terms-of-trade is separating: each agent has the same equilibrium utility, matching probability, expected fee paid, and partner distribution.*

Proof of Lemma 6. Let $f(i,j) \equiv u^a(i,j) + u^b(j,i)$. Fix equilibrium utilities U^a, U^b and consider any terms-of-trade $(\phi^s, \lambda^s, G^s)_{s=a,b}$ offered in equilibrium. The participation constraint binds at every type in supp(G^s), for otherwise the platform could raise ϕ^s at that type. Thus eliminating fees reduces the objective of (15) to

$$m(\lambda^a, \lambda^b) \iint \left(f(i,j) - \frac{U^a(i)}{\lambda^a} - \frac{U^b(j)}{\lambda^b} \right) dG^a(i)\, dG^b(j).$$

This is linear in $G^a \times G^b$, so optimality requires the integrand to be constant on $\text{supp}(G^a) \times \text{supp}(G^b)$, with common value V^*.

Now compare V^* to $V^O(i,j)$, the value of the best *separating* terms-of-trade attracting (i,j):

$$V^O(i,j) = \max_{\lambda^a,\lambda^b} m(\lambda^a,\lambda^b)\left(f(i,j) - \frac{U^a(i)}{\lambda^a} - \frac{U^b(j)}{\lambda^b}\right).$$

The pooling terms-of-trade uses a single (λ^a,λ^b) for all pairs in its support, whereas $V^O(i,j)$ optimizes freely, so $V^* \leq V^O(i,j)$ for every such pair, with equality if and only if (λ^a,λ^b) solves the maximization defining $V^O(i,j)$.

Since m is strictly concave, the maximizer for $V^O(i,j)$ is unique. If $V^* < V^O(i,j)$ for some (i,j) in the support, a platform profitably deviates by offering the separating terms-of-trade with value $V^O(i,j)$, contradicting equilibrium. The remaining case is $V^* = V^O(i,j)$ for every pair in the support. Replace the pooling terms-of-trade with a collection of separating ones, one for each $(i,j) \in \text{supp}(G^a) \times \text{supp}(G^b)$ in measure $dG^a(i)\,dG^b(j)$, each with the same (λ^a,λ^b) and the corresponding fees. Agents face the same matching probabilities, fees, partner distribution, and equilibrium utility, and platforms continue to break even. ■

With separation, problem (15) decomposes into two steps: (1) for each pair (k^a,k^b), choose $(\lambda^s,\phi^s)_{s=a,b}$ to maximize the platform's value, and (2) choose (k^a,k^b) to maximize that value. Eliminating the fees through the binding participation constraints in step 1 gives the platform's value from attracting (k^a,k^b):

$$V^O(k^a,k^b) = \max_{(\lambda^a,\lambda^b)\in\mathbb{\Lambda}} m(\lambda^a,\lambda^b)\left(u^a(k^a,k^b) + u^b(k^b,k^a) - \frac{U^a(k^a)}{\lambda^a} - \frac{U^b(k^b)}{\lambda^b}\right). \tag{16}$$

In equilibrium, free entry gives $V^O(k^a,k^b) = c$ at every active terms-of-trade and $V^O(k^a,k^b) \leq c$ elsewhere.

The general sorting result follows directly from supermodularity of V^O:

Lemma 7 *If $V^O(k^a,k^b)$ is supermodular (submodular) in (k^a,k^b), any CSE with observable types exhibits PAM (NAM).*

Proof of Lemma 7. Suppose V^O is supermodular and, toward a contradiction, that there are two active terms-of-trade attracting (k_1^a,k_2^b) and (k_2^a,k_1^b) with $k_1^a > k_2^a$ and $k_1^b > k_2^b$. Free entry gives $V^O(k_1^a,k_2^b) = V^O(k_2^a,k_1^b) = c$, while supermodularity gives $V^O(k_1^a,k_1^b) + V^O(k_2^a,k_2^b) > V^O(k_1^a,k_2^b) + V^O(k_2^a,k_1^b) = 2c$. Thus $V^O(k_1^a,k_1^b) > c$ or $V^O(k_2^a,k_2^b) > c$, contradicting the optimality of the active terms-of-trade. The submodular case is symmetric. ■

Checking the supermodularity hypothesis of Lemma 7 directly can be awkward because V^O is defined as the value of (16) and includes the endogenous equilibrium utility. The CES matching function used throughout the body yields a closed form that delivers a more tractable sufficient condition in terms of the total match surplus alone.

Proposition 5 *Suppose* $m(\lambda^a, \lambda^b) = (1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma)^{1/\gamma}$*. Let* $f(i,j) \equiv u^a(i,j) + u^b(j,i)$ *denote total match surplus. Any CSE with observable types exhibits PAM (NAM) whenever the set* $\{(i,j) : f(i,j) > 0\}$ *is a sublattice and* $f^{\frac{\gamma}{1+\gamma}}$ *is supermodular (submodular) on that set.*

Proof of Proposition 5. Solving (16) gives $\lambda^s = (U^s/f(k^a,k^b))^{1/(1+\gamma)}$. Plugging this into equation (16) gives

$$V^O(k^a,k^b)^{\frac{\gamma}{1+\gamma}} = f(k^a,k^b)^{\frac{\gamma}{1+\gamma}} - U^a(k^a)^{\frac{\gamma}{1+\gamma}} - U^b(k^b)^{\frac{\gamma}{1+\gamma}}. \tag{17}$$

Since the right-hand side is additively separable in the U terms, supermodularity of $f^{\frac{\gamma}{1+\gamma}}$ on $\{f > 0\}$ implies supermodularity of $V^{O\frac{\gamma}{1+\gamma}}$ on the same set. The two active terms-of-trade lie in $\{f > 0\}$ by free entry, and the sublattice property ensures that the recombined points (k_1^a, k_1^b) and (k_2^a, k_2^b) also lie in $\{f > 0\}$. The rearrangement argument in the proof of Lemma 7 then applies to $V^{O\frac{\gamma}{1+\gamma}}$: at any two negatively-sorted active terms-of-trade, $V^O(k_1^a,k_2^b)^{\frac{\gamma}{1+\gamma}} = V^O(k_2^a,k_1^b)^{\frac{\gamma}{1+\gamma}} = c^{\frac{\gamma}{1+\gamma}}$, and supermodularity of $V^{O\frac{\gamma}{1+\gamma}}$ gives $V^O(k_1^a,k_1^b)^{\frac{\gamma}{1+\gamma}} + V^O(k_2^a,k_2^b)^{\frac{\gamma}{1+\gamma}} > 2c^{\frac{\gamma}{1+\gamma}}$. Since $x \mapsto x^{\frac{1+\gamma}{\gamma}}$ is strictly increasing, this implies $V^O(k_1^a,k_1^b) > c$ or $V^O(k_2^a,k_2^b) > c$, the desired contradiction. The submodular case is symmetric. ∎

We illustrate with the two parametric environments from the body.

CES surplus. Specialize to the symmetric CES payoffs of equation (11) in Section 5, so $f(i,j) = 2\left(\frac{1}{2}i^{\frac{\theta-1}{\theta}} + \frac{1}{2}j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$. Direct computation gives

$$\operatorname{sign}\left(\frac{\partial^2}{\partial k^a \partial k^b} f(k^a,k^b)^{\frac{\gamma}{1+\gamma}}\right) = \operatorname{sign}(1+\gamma-\theta),$$

so $f^{\frac{\gamma}{1+\gamma}}$ is supermodular when $\theta < 1+\gamma$ and submodular when $\theta > 1+\gamma$. Proposition 5 thus gives PAM when $\theta < 1+\gamma$ and NAM when $\theta > 1+\gamma$. The threshold is the same one that appears in Section 5.2 for the private-information CSE.

Disease surplus. Assume $u^s(i,j) = 1 - \kappa\, i(1-j)$, the disease-transmission payoffs from Section 6, so $f(i,j) = 2 - \kappa(i + j - 2ij)$. The viable region $\{(i,j) : f(i,j) > 0\}$ is a sublattice: using $(u,v) = (1-2i, 1-2j)$, this region equals $\{(u,v) \in [-1,1]^2 : uv > (\kappa - 4)/\kappa\}$, a sublattice of $[-1,1]^2$ when $\kappa \le 4$. The cross-partial of $f^{\frac{\gamma}{1+\gamma}}$ is

$$\frac{\partial^2}{\partial i\, \partial j} f(i,j)^{\frac{\gamma}{1+\gamma}} = \gamma(1+\gamma)^{-2}\kappa\, f(i,j)^{-\frac{1}{1+\gamma}} \left(\frac{4-\kappa}{f(i,j)} + 2\gamma \right).$$

For $\kappa < 4$, this is positive on the viable region $\{f > 0\}$, and Proposition 5 delivers PAM. When $\gamma < 1$, this observable-type condition is strictly weaker than the private-information condition $\kappa \le \min\{2(1+\gamma), 4\}$ from Proposition 3.

B Proofs for Section 4

B.1 Partial Equilibrium

Proof of Lemma 1. First, we verify that the construction yields a partial equilibrium. We check each condition of Definition 1:

1. (Optimal Search) Take any $\tau \in T^p$. By construction:

   (a) $U^s(i) \ge \bar{U}^s(i, \tau, \Lambda^s(\tau))$ for all $i \in \mathbb{I}^s$, since this is exactly the first constraint of problem (6);

   (b) $\int_{\mathbb{I}^s} U^s(i) dG^s(i) = \int_{\mathbb{I}^s} \bar{U}^s(i, \tau, \Lambda^s(\tau)) dG^s(i)$, from the second constraint.

   Conversely, for any $\tau \notin T^p$, by construction there exist no $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ satisfying the constraints of problem (6), hence none satisfying conditions 1(a) and 1(b) of Definition 1.

2. (Profit Maximization) Take any $\tau \in T^p$ with $\tau \notin T$. By construction, $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ does not solve problem (6). Therefore, there exists some $(\hat{\lambda}^s, \hat{\phi}^s, \hat{G}^s)_{s=a,b}$ satisfying the constraints with strictly higher objective value. Let $\hat{\tau} = (\hat{\phi}^s, \hat{G}^s)_{s=a,b}$, so $\Lambda^s(\hat{\tau}) = \hat{\lambda}^s$. Then $\hat{\tau} \in T^p$ and $V(\hat{\tau}, \Lambda(\hat{\tau})) > V(\tau, \Lambda(\tau))$.

For the converse, take any partial equilibrium $\{T^p, T, \Lambda, U\}$ and any $\tau \in T$. We must show that $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ solves problem (6). From the Optimal Search condition in Definition 1, this tuple satisfies the constraints of (6). Suppose it is not optimal: there exists a tuple $(\hat{\lambda}^s, \hat{\phi}^s, \hat{G}^s)_{s=a,b}$ satisfying the constraints of (6) with

$$m(\hat{\lambda}^a, \hat{\lambda}^b)(\hat{\phi}^a + \hat{\phi}^b) \;>\; m(\lambda^a, \lambda^b)(\phi^a + \phi^b) \;\ge\; 0,$$

where the second inequality holds because $m \geq 0$ and feasible terms-of-trade have $\phi^a + \phi^b$ nonnegative. In particular $\hat{\phi}^a + \hat{\phi}^b > 0$. Let $\hat{\tau} = (\hat{\phi}^s, \hat{G}^s)_{s=a,b}$.

Case 1: Optimal Search pins down Λ. On each side s, Optimal Search uniquely determines $\Lambda^s(\hat{\tau}) = \hat{\lambda}^s$ whenever either $U^s(i) > 0$ for some $i \in \text{supp}(\hat{G}^s)$; or $U^s(i) = 0$ for all $i \in \text{supp}(\hat{G}^s)$ and $\hat{\phi}^s < \int u^s(i,j)\, d\hat{G}^{\bar{s}}(j)$ for some $i \in \text{supp}(\hat{G}^s)$: in the first subcase, Optimal Search determines $\Lambda^s(\hat{\tau}) > 0$, while in the second it forces $\Lambda^s(\hat{\tau}) = 0$. If this holds on both sides, Definition 1 gives $\Lambda(\hat{\tau}) = \hat{\lambda}$, so $\bar{V}(\hat{\tau}) = V(\hat{\tau}, \hat{\lambda}) > \bar{V}(\tau)$ and Profit Maximization contradicts $\tau \in T$.

Case 2: Optimal Search does not uniquely determine Λ. On at least one side s, both $U^s(i) = 0$ and $\hat{\phi}^s = \int u^s(i,j)\, d\hat{G}^{\bar{s}}(j)$ for all $i \in \text{supp}(\hat{G}^s)$. $V(\hat{\tau}, \cdot)$ is strictly decreasing in λ^s (fixed positive sum of fees and m strictly decreasing in each argument), so without loss of generality we take $\hat{\lambda}^s = 0$ on every degenerate side.

Construct a perturbed terms-of-trade $\tilde{\tau}$ by reducing $\hat{\phi}^s$ to $\tilde{\phi}^s = \hat{\phi}^s - \epsilon$ on each degenerate side, with $\epsilon \in (0, \frac{1}{2}(\hat{\phi}^a + \hat{\phi}^b))$, leaving $\hat{G}^a, \hat{G}^b$ unchanged. We verify $\tilde{\tau} \in T^p$, $\Lambda^s(\tilde{\tau}) = 0$ on a perturbed side s, and $\Lambda^{\bar{s}}(\tilde{\tau}) = \Lambda^{\bar{s}}(\hat{\tau})$ on an unperturbed side $\bar{s}$. On any perturbed side(s), Condition 1(a) in Definition 1 evaluated at $i \in \text{supp}(\hat{G}^s)$ implies $\Lambda^s(\tilde{\tau}) = 0$. On the unperturbed side (if any), the side is non-degenerate by choice, so Case 1's uniqueness applies at $\hat{\tau}$ and the constraints at $\tilde{\tau}$ coincide with those at $\hat{\tau}$, so $\Lambda^{\bar{s}}(\tilde{\tau}) = \Lambda^{\bar{s}}(\hat{\tau}) = \hat{\lambda}^{\bar{s}}$. Therefore

$$\bar{V}(\tilde{\tau}) \;=\; V(\tilde{\tau}, \hat{\lambda}) \;=\; V(\hat{\tau}, \hat{\lambda}) - m(\hat{\lambda})\, \epsilon',$$

where ϵ' equals ϵ (one-sided degeneracy) or 2ϵ (two-sided). Since $V(\hat{\tau}, \hat{\lambda}) - \bar{V}(\tau) > 0$ is a fixed positive gap independent of ϵ, for ϵ sufficiently small $\bar{V}(\tilde{\tau}) > \bar{V}(\tau)$, contradicting $\tau \in T$ by Profit Maximization. ■

B.2 Characterization of CSE

Before turning to the proof of Proposition 1, we establish a few preliminary results. First, we show that equilibrium utility is continuous:

Lemma 8 *In any CSE, U^s is continuous for $s \in \{a, b\}$.*

Proof of Lemma 8. Suppose not, so U^s is discontinuous at some point $i^* \in \mathbb{I}^s$. Then there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exist a point $i \in \mathbb{I}^s$ with $|i - i^*| < \delta$ and $|U^s(i) - U^s(i^*)| > \epsilon$.

First assume it is possible to find such a point with $U^s(i^*) - U^s(i) > \epsilon$. By the third condition in the definition of competitive search equilibrium, type i^* must obtain utility $U^s(i^*) = \Lambda^s(\tau)(\int u^s(i^*, j) dG^{\bar{s}}(j) - \phi^s)$ at some terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b} \in T$.

Continuity of u^s in its first argument implies we can choose δ sufficiently small such that $\int u^s(i^*, j)dG^{\bar{s}}(j) - \int u^s(i, j)dG^{\bar{s}}(j) < \epsilon$. Since $0 \leq \Lambda^s(\tau) \leq 1$, this means

$$U^s(i) < U^s(i^*) - \epsilon = \Lambda^s(\tau)\left(\int u^s(i^*, j)dG^{\bar{s}}(j) - \phi^s\right) - \epsilon < \Lambda^s(\tau)\left(\int u^s(i, j)dG^{\bar{s}}(j) - \phi^s\right).$$

The first inequality comes from the assumed discontinuity in U^s. The equation is the indifference condition of i^*. The second inequality uses the continuity of u^s. But this implies that the terms-of-trade τ do not satisfy condition 1(a) in the definition of partial equilibrium for the type i agent, a contradiction.

If instead we have $U^s(i) - U^s(i^*) > \epsilon$, we reverse the role of i and i^* in the proof, but the argument is otherwise unchanged. ■

Second, the definition of equilibrium determines how the agents' trading probability Λ^s depends on the terms-of-trade τ. We consider the inverse mapping: find the terms-of-trade that maximizes the platform fees given agent trading probabilities and the agents on the other side of the market, while respecting participation and incentive constraints. We call this the best response set.

Lemma 9 *Assume Supermodularity. Fix continuous nonnegative equilibrium utilities U^a, U^b and matching probabilities $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ with $\lambda^a, \lambda^b > 0$. For each side $s \in \{a, b\}$ and each partner type $k^{\bar{s}} \in \mathbb{I}^{\bar{s}}$, define the side-s best-response set*

$$\Omega^s(k^{\bar{s}}) = \arg\max_{k' \in \mathbb{I}^s}\left[u^s(k', k^{\bar{s}}) - \frac{U^s(k')}{\lambda^s}\right],$$

and let $\omega^s(k^{\bar{s}})$ be the largest element of $\Omega^s(k^{\bar{s}})$. Let $\omega(k^a, k^b) = (\omega^a(k^b), \omega^b(k^a))$. Then:

1. *ω is well-defined and non-decreasing on $\mathbb{I}^a \times \mathbb{I}^b$.*

2. *For any $(\underline{k}^a, \underline{k}^b) \in \mathbb{I}^a \times \mathbb{I}^b$, let $S = \{(k^a, k^b) \in \mathbb{I}^a \times \mathbb{I}^b : k^a \geq \underline{k}^a, k^b \geq \underline{k}^b\}$. If $\omega(\underline{k}^a, \underline{k}^b) \geq (\underline{k}^a, \underline{k}^b)$, then ω has a fixed point in S.*

3. *If $(\hat{k}^a, \hat{k}^b)$ is such a fixed point, set $\hat{\phi}^s = u^s(\hat{k}^s, \hat{k}^{\bar{s}}) - U^s(\hat{k}^s)/\lambda^s$. Then for each s and every $k' \in \mathbb{I}^s$, $U^s(k') \geq \lambda^s(u^s(k', \hat{k}^{\bar{s}}) - \hat{\phi}^s)$, with equality at $k' = \hat{k}^s$. This gives full incentive compatibility (upward and downward) and participation; in particular, $(\lambda^s, \hat{\phi}^s, \hat{k}^s)$ satisfies the constraints of problem (7), including agent rationality.*

Proof of Lemma 9. *Part 1.* Continuity of u^s and U^s with compact $\mathbb{I}^s$ gives a nonempty compact argmax set. Supermodularity of u^s gives increasing differences in $(k', k^{\bar{s}})$, so the largest-element selection $\omega^s(k^{\bar{s}})$ is non-decreasing in $k^{\bar{s}}$ by Topkis's Theorem.

Part 2. ω is a non-decreasing function mapping the complete sublattice S into itself (by hypothesis $\omega(\underline{k}^a, \underline{k}^b) \geq (\underline{k}^a, \underline{k}^b)$ and monotonicity), so it has a fixed point by Tarski's Fixed Point Theorem.

Part 3. The fixed-point condition states $\hat{k}^s$ maximizes $u^s(k', \hat{k}^{\bar{s}}) - U^s(k')/\lambda^s$ over $\mathbb{I}^s$, which rearranges to $U^s(k') \geq \lambda^s(u^s(k', \hat{k}^{\bar{s}}) - \hat{\phi}^s)$ for all $k' \in \mathbb{I}^s$, with equality at $\hat{k}^s$. This gives full incentive compatibility and participation. The agent rationality constraint $\hat{\phi}^s \leq u^s(\hat{k}^s, \hat{k}^{\bar{s}})$ follows from $U^s(\hat{k}^s) \geq 0$ and $\lambda^s > 0$. ■

The next lemma packages a pattern used repeatedly in the proofs that follow: given a starting point where the sublattice condition holds, it produces a feasible separating terms-of-trade in T^p whose fees are at least those at the starting point.

Lemma 10 *Make all the hypotheses of Lemma 9 and assume Common Ranking. Suppose* $(\underline{k}^a, \underline{k}^b) \in \mathbb{I}^a \times \mathbb{I}^b$ *satisfies the sublattice condition* $\omega(\underline{k}^a, \underline{k}^b) \geq (\underline{k}^a, \underline{k}^b)$. *Then* ω *has a fixed point* $(\hat{k}^a, \hat{k}^b) \geq (\underline{k}^a, \underline{k}^b)$, *and setting* $\hat{\phi}^s = u^s(\hat{k}^s, \hat{k}^{\bar{s}}) - U^s(\hat{k}^s)/\lambda^s$, *the separating terms-of-trade* $\hat{\tau} = (\hat{\phi}^a, \hat{\phi}^b, \delta_{\hat{k}^a}, \delta_{\hat{k}^b})$ *satisfies the constraints of problem (7), lies in* T^p *with* $\Lambda(\hat{\tau}) = (\lambda^a, \lambda^b)$, *and has fees bounded below by the starting-point values:*

$$\hat{\phi}^s \geq u^s(\underline{k}^s, \underline{k}^{\bar{s}}) - \frac{U^s(\underline{k}^s)}{\lambda^s}, \qquad s \in \{a, b\}, \tag{18}$$

with strict inequality whenever $\hat{k}^{\bar{s}} > \underline{k}^{\bar{s}}$ *and* $\underline{k}^s > \underline{i}^s$.

Proof of Lemma 10. Parts 2 and 3 of Lemma 9 deliver the fixed point $(\hat{k}^a, \hat{k}^b)$ and the fact that $(\lambda^s, \hat{\phi}^s, \hat{k}^s)_{s=a,b}$ satisfies the constraints of (7), including full (upward and downward) incentive compatibility, participation, and agent rationality. With G^s degenerate at $\hat{k}^s$, these constraints coincide with the Optimal Search conditions of Definition 1 at rates (λ^a, λ^b), so $\hat{\tau} \in T^p$ and $\Lambda(\hat{\tau}) = (\lambda^a, \lambda^b)$.

For the fee bound, optimality of $\hat{k}^s$ in $\Omega^s(\hat{k}^{\bar{s}})$ gives

$$\hat{\phi}^s = u^s(\hat{k}^s, \hat{k}^{\bar{s}}) - \frac{U^s(\hat{k}^s)}{\lambda^s} \geq u^s(\underline{k}^s, \hat{k}^{\bar{s}}) - \frac{U^s(\underline{k}^s)}{\lambda^s}.$$

Since $\hat{k}^{\bar{s}} \geq \underline{k}^{\bar{s}}$, Common Ranking gives $u^s(\underline{k}^s, \hat{k}^{\bar{s}}) \geq u^s(\underline{k}^s, \underline{k}^{\bar{s}})$. When $\hat{k}^{\bar{s}} > \underline{k}^{\bar{s}}$ and $\underline{k}^s > \underline{i}^s$, Common Ranking and Supermodularity imply $u^s(\underline{k}^s, \cdot)$ is strictly increasing on $\mathbb{I}^{\bar{s}}$, giving strict inequality. ■

Proof of Proposition 1. Let $\underline{i}^s$ and $\bar{i}^s$ denote the lower and upper endpoints of the compact set $\mathbb{I}^s$.

Part 1. Take any $\tau = (\phi^s, G^s)_{s=a,b} \in T$, and let $\lambda^s = \Lambda^s(\tau)$. Market Clearing in Definition 2 bounds the total side-s agent flow: $\nu^s(T) \le I^s < \infty$. Since ν^s is a finite Radon measure on T and $d\mu = \Lambda^s(\tau)\, d\nu^s$ (Consistency), $\mu(\{\tau \in T : \Lambda^s(\tau) = 0\}) = \int_{\{\Lambda^s=0\}} \Lambda^s(\tau)\, d\nu^s(\tau) = 0$ for each s. Hence $\lambda^s = \Lambda^s(\tau) > 0$ at μ-a.e. $\tau \in T$ on each side.

By Lemma 1 and Free Entry, $(\lambda^s, \phi^s, G^s)_{s=a,b}$ solves problem (6) with maximal value c. Suppose, toward contradiction, that τ is not separating in the sense of Definition 3: for some $i, i' \in \text{supp}(G^a)$ and $j \in \text{supp}(G^b)$, $u^b(j,i) > u^b(j,i')$. (The case where only G^b is nondegenerate is symmetric). Let $k_1^s = \max(\text{supp}(G^s))$. Common Ranking and Supermodularity together imply that $u^s(k,\cdot)$ is strictly increasing on $\mathbb{I}^{\bar{s}}$ at every $k > \underline{i}^s$. Thus $u^b(k_1^b, k_1^a) > u^b(k_1^b, i)$ for some $i \in \text{supp}(G^a)$.

The (P) participation constraint at τ is in integrated form, but combined with the pointwise inequality from Optimal Search 1(a) it forces equality G^s-a.e. on $\text{supp}(G^s)$. Since U^s is continuous (Lemma 8) and u^s is continuous, the gap $i \mapsto U^s(i)/\lambda^s - \int u^s(i,j)\, dG^{\bar{s}}(j) + \phi^s$ is continuous, vanishes G^s-a.e. on $\text{supp}(G^s)$, and hence vanishes on all of $\text{supp}(G^s)$, in particular at k_1^s. We use this pointwise binding participation at k_1^s in what follows.

Positive-m case. Suppose $m(\lambda^a, \lambda^b) > 0$. Apply Lemma 9 with matching probabilities (λ^a, λ^b) and sublattice $S = \{(k^a, k^b) \in \mathbb{I}^a \times \mathbb{I}^b : k^a \ge k_1^a, k^b \ge k_1^b\}$. We verify the sublattice condition $\omega(k_1^a, k_1^b) \ge (k_1^a, k_1^b)$ on each side. On side b: for any $k < k_1^b$, define

$$\Delta^b(k) \equiv \left[u^b(k_1^b, k_1^a) - \frac{U^b(k_1^b)}{\lambda^b}\right] - \left[u^b(k, k_1^a) - \frac{U^b(k)}{\lambda^b}\right].$$

The binding participation constraint at k_1^b gives $U^b(k_1^b)/\lambda^b = \int_{\mathbb{I}^a} u^b(k_1^b, i)\, dG^a(i) - \phi^b$, and Optimal Search at k gives $U^b(k)/\lambda^b \ge \int_{\mathbb{I}^a} u^b(k, i)\, dG^a(i) - \phi^b$. Substituting,

$$\Delta^b(k) \ge \int_{\mathbb{I}^a} \Big(\big[u^b(k_1^b, k_1^a) - u^b(k, k_1^a)\big] - \big[u^b(k_1^b, i) - u^b(k, i)\big] \Big)\, dG^a(i).$$

Since $k_1^b > k$ and $k_1^a \ge i$ for all $i \in \text{supp}(G^a)$ with strict inequality on a set of positive G^a-measure (non-degeneracy), Supermodularity makes the integrand strictly positive on that set. Hence $\Delta^b(k) > 0$, establishing $\max \Omega^b(k_1^a) \ge k_1^b$.

On side a: the same argument applies, with G^b playing the role of G^a. Since G^b may be degenerate at k_1^b, Supermodularity gives only $\Delta^a(k) \ge 0$, establishing $\max \Omega^a(k_1^b) \ge k_1^a$.

By Lemma 10, ω has a fixed point $(\hat{k}^a, \hat{k}^b) \in S$, and the associated $\hat{\tau} \in T^p$ satisfies the constraints of problem (7) with fees satisfying $\hat{\phi}^s \ge u^s(k_1^s, k_1^{\bar{s}}) - U^s(k_1^s)/\lambda^s$ for $s \in \{a, b\}$.

Substituting the binding participation $U^s(k_1^s)/\lambda^s = \int_{\mathbb{I}^{\bar{s}}} u^s(k_1^s, i)\, dG^{\bar{s}}(i) - \phi^s$ at τ gives

$$\hat{\phi}^s \geq u^s(k_1^s, k_1^{\bar{s}}) - \int_{\mathbb{I}^{\bar{s}}} u^s(k_1^s, i)\, dG^{\bar{s}}(i) + \phi^s.$$

Common Ranking gives $\hat{\phi}^a \geq \phi^a$ (weakly). On side b, $u^b(k_1^b, k_1^a) > \int u^b(k_1^b, j)\, dG^a(j)$ by supposition, so $\hat{\phi}^b > \phi^b$. Thus $\hat{\phi}^a + \hat{\phi}^b > \phi^a + \phi^b \geq 0$. Since $m(\lambda^a, \lambda^b) > 0$, the deviation yields $\bar{V}(\hat{\tau}) > c$, contradicting Free Entry.

Zero-m case. Suppose instead $m(\lambda^a, \lambda^b) = 0$, as may be the case when $c = 0$. Since $\mathbb{\Lambda}$ is the closure of $\mathbb{\Lambda}^o$ and is a down set, and since $\lambda^b > 0$, there exist rates $(\lambda^a, \hat{\lambda}^b) \in \mathbb{\Lambda}^o$ with $\hat{\lambda}^b \in (0, \lambda^b)$ arbitrarily close to λ^b.

Apply Lemma 9 at rates $(\lambda^a, \hat{\lambda}^b)$ on the same sublattice S. On side a, the sublattice condition $\omega^a(k_1^b) \geq k_1^a$ holds unchanged from the Positive-m case, since λ^a is unchanged. On side b, we show the condition holds for $\hat{\lambda}^b$ sufficiently close to λ^b.

At the perturbed rate, define $\Delta^{b,\hat{\lambda}^b}(k)$ analogously to $\Delta^b(k)$ in the positive-m case, with λ^b replaced by $\hat{\lambda}^b$. The binding participation constraint at k_1^b and Optimal Search at k (both from the original τ, at rate λ^b) yield

$$\frac{U^b(k_1^b)}{\hat{\lambda}^b} = \frac{\lambda^b}{\hat{\lambda}^b}\left[\int_{\mathbb{I}^a} u^b(k_1^b, i)\, dG^a(i) - \phi^b\right], \qquad \frac{U^b(k)}{\hat{\lambda}^b} \geq \frac{\lambda^b}{\hat{\lambda}^b}\left[\int_{\mathbb{I}^a} u^b(k, i)\, dG^a(i) - \phi^b\right].$$

Substituting and writing $\lambda^b/\hat{\lambda}^b = 1 + (\lambda^b/\hat{\lambda}^b - 1)$,

$$\begin{aligned}\Delta^{b,\hat{\lambda}^b}(k) \geq \int_{\mathbb{I}^a} \Big(\big[u^b(k_1^b, k_1^a) - u^b(k, k_1^a)\big] - \big[u^b(k_1^b, i) - u^b(k, i)\big]\Big)\, dG^a(i) \\ - \left(\frac{\lambda^b}{\hat{\lambda}^b} - 1\right)\int_{\mathbb{I}^a} \big[u^b(k_1^b, i) - u^b(k, i)\big]\, dG^a(i).\end{aligned}$$

Denote the two integrals $D(k)$ and $E(k)$ respectively; neither depends on $\hat{\lambda}^b$. As in the Positive-m case, Supermodularity and non-degeneracy of G^a give $D(k) > 0$ for all $k < k_1^b$. If $E(k) \leq 0$, then $\Delta^{b,\hat{\lambda}^b}(k) \geq D(k) > 0$ trivially. Thus focus on k with $E(k) > 0$.

We show $\inf\{D(k)/E(k) : k < k_1^b,\ E(k) > 0\} > 0$. Suppose not: there exists a sequence $k_n \in \mathbb{I}^b$ with $k_n < k_1^b$, $E(k_n) > 0$, and $D(k_n)/E(k_n) \to 0$. By compactness of $\mathbb{I}^b$, pass to a subsequence (still denoted k_n) with $k_n \to k^* \in \mathbb{I}^b$, $k^* \leq k_1^b$.

If $k^* < k_1^b$, then by continuity $D(k_n) \to D(k^*)$ and $E(k_n) \to E(k^*)$, with $D(k^*) > 0$ by Supermodularity and non-degeneracy of G^a. If also $E(k^*) > 0$, then $D/E \to D(k^*)/E(k^*) > 0$, contradicting $D/E \to 0$. If instead $E(k^*) = 0$, then $D(k_n)/E(k_n) \to \infty$, again a contradiction. Hence $k^* = k_1^b$, and in particular k_1^b is an accumulation point of $\mathbb{I}^b$ from below.

Fix $i^* < k_1^a$ in $\mathbb{I}^a$ with $G^a(i^*) > 0$ (exists by non-degeneracy of G^a). By Supermodularity, the integrand of D is non-increasing in i and non-negative for $i \le k_1^a$, so

$$D(k_n) \ge G^a(i^*) \cdot \Big(\big[u^b(k_1^b, k_1^a) - u^b(k_n, k_1^a)\big] - \big[u^b(k_1^b, i^*) - u^b(k_n, i^*)\big] \Big),$$

and the integrand of E is non-decreasing in i and bounded above by its value at k_1^a, so $E(k_n) \le u^b(k_1^b, k_1^a) - u^b(k_n, k_1^a)$.

Partition the indices as $A = \{n : u^b(k_1^b, i^*) \le u^b(k_n, i^*)\}$ and $B = \mathbb{N} \setminus A$. For $n \in A$, the bracketed term subtracted in the D bound is non-positive, so $D(k_n) \ge G^a(i^*) \cdot [u^b(k_1^b, k_1^a) - u^b(k_n, k_1^a)] \ge G^a(i^*) \cdot E(k_n)$, giving $D(k_n)/E(k_n) \ge G^a(i^*) > 0$ for every $n \in A$. This contradicts $D/E \to 0$ unless A is finite.

So A is finite; drop those indices. The remaining sequence is $k_n \uparrow k_1^b$ in $\mathbb{I}^b$ with $u^b(k_1^b, i^*) > u^b(k_n, i^*)$ for all n. Define

$$r_n \equiv \frac{u^b(k_1^b, k_1^a) - u^b(k_n, k_1^a)}{u^b(k_1^b, i^*) - u^b(k_n, i^*)}.$$

Both numerator and denominator are positive, and Limit Supermodularity (with $i = k_1^b$, $j' = i^*$, $j = k_1^a$) gives $\liminf_n r_n = R > 1$. The D and E bounds combine to

$$\frac{D(k_n)}{E(k_n)} \ge G^a(i^*) \cdot \left(1 - \frac{1}{r_n}\right),$$

so $\liminf_n D(k_n)/E(k_n) \ge G^a(i^*) \cdot (1 - 1/R) > 0$, again contradicting $D/E \to 0$.

This establishes the uniform lower bound $D(k)/E(k) \ge \eta$ on $\{k < k_1^b : E(k) > 0\}$, where $\eta \equiv G^a(i^*)(1 - 1/R) > 0$. Any $\hat{\lambda}^b \in (\lambda^b/(1+\eta), \lambda^b)$ thus implies $D(k) > (\lambda^b/\hat{\lambda}^b - 1)E(k)$, hence $\Delta^{b,\hat{\lambda}^b}(k) > 0$, for all $k < k_1^b$ with $E(k) > 0$. Combined with the trivial bound when $E(k) \le 0$, we obtain $\Delta^{b,\hat{\lambda}^b}(k) > 0$ for all $k < k_1^b$, establishing the sublattice condition.

By Lemma 10 at rates $(\lambda^a, \hat{\lambda}^b)$, the resulting $\hat{\tau} \in T^p$ has fees bounded by $\hat{\phi}^s \ge u^s(k_1^s, k_1^{\bar{s}}) - U^s(k_1^s)/\hat{\lambda}^s$, where $\hat{\lambda}^a = \lambda^a$. Substituting binding participation at τ,

$$\hat{\phi}^b \ge u^b(k_1^b, k_1^a) - \frac{\lambda^b}{\hat{\lambda}^b}\left[\int_{\mathbb{I}^a} u^b(k_1^b, i)\, dG^a(i) - \phi^b\right], \quad \hat{\phi}^a \ge u^a(k_1^a, k_1^b) - \int_{\mathbb{I}^b} u^a(k_1^a, i)\, dG^b(i) + \phi^a.$$

At $\hat{\lambda}^b = \lambda^b$, the first bound equals $u^b(k_1^b, k_1^a) - \int u^b(k_1^b, i)\, dG^a(i) + \phi^b$, which is strictly greater than ϕ^b by supposition. The bound is continuous in $\hat{\lambda}^b$, so for $\hat{\lambda}^b$ in some left-neighborhood of λ^b, it remains strictly above ϕ^b. Picking $\hat{\lambda}^b$ in the intersection of this neighborhood and $(\lambda^b/(1+\eta), \lambda^b)$ gives $\hat{\phi}^b > \phi^b$. The second bound gives $\hat{\phi}^a \ge \phi^a$ regardless. Thus $\hat{\phi}^a + \hat{\phi}^b > \phi^a + \phi^b \ge 0$. Since $m(\lambda^a, \lambda^b) > 0$, the deviation yields $\bar{V}(\hat{\tau}) > c$, contradicting Free Entry.

This establishes that any CSE is separating in the sense of Definition 3. Moreover, at most one of G^a, G^b is non-degenerate at μ-a.e. $\tau \in T$: if both were non-degenerate, then $k_1^s > \underline{i}^s$ on each side, so $u^b(k_1^b, \cdot)$ would be strictly increasing on $\mathbb{I}^a$ and $u^b(k_1^b, k_1^a) > u^b(k_1^b, j)$ for any $j < k_1^a$ in $\text{supp}(G^a)$, contradicting separation on side b. At any such τ, picking $k^s \in \text{supp}(G^s)$ on each side, the indifference of $u^s(k^s, \cdot)$ on $\text{supp}(G^{\bar{s}})$ reduces the (P) constraints at τ to the (PS) constraints at $(\lambda^s, \phi^s, k^s)_{s=a,b}$; Free Entry gives value c.

It remains to show no (PS)-feasible tuple attains value greater than c. Take any (PS)-feasible $(\lambda^a, \lambda^b, \phi^a, \phi^b, k^a, k^b)$. If $m(\lambda^a, \lambda^b) = 0$, its value is $0 \leq c$. Otherwise $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}^o$ with $\lambda^a, \lambda^b > 0$. Apply Lemma 10 at rates (λ^a, λ^b) with starting point (k^a, k^b): by the (PS) downward IC and binding participation at k^s, any $i < k^s$ satisfies $u^s(i, k^{\bar{s}}) - U^s(i)/\lambda^s \leq \phi^s = u^s(k^s, k^{\bar{s}}) - U^s(k^s)/\lambda^s$, so $\omega^s(k^{\bar{s}}) \geq k^s$ on each side. The lemma yields $\hat{\tau} \in T^p$ with fees $\hat{\phi}^s \geq u^s(k^s, k^{\bar{s}}) - U^s(k^s)/\lambda^s = \phi^s$, so $\bar{V}(\hat{\tau}) \geq m(\lambda^a, \lambda^b)(\phi^a + \phi^b)$. Finally, Free Entry implies $c \geq \bar{V}(\hat{\tau})$, so the (PS) value of the original tuple $(\lambda^a, \lambda^b, \phi^a, \phi^b, k^a, k^b)$ is at most c.

Part 2. Assume the hypotheses (i), (ii), (iii).

Step 1: $T \subseteq T^p$. Fix $\tau \in T$ with associated $(\lambda^s, \phi^s, k^s)_{s=a,b}$ as in (i). We verify Optimal Search conditions 1(a) and 1(b) at rates (λ^a, λ^b). Condition 1(a) is $U^s(i) \geq \bar{U}^s(i, \tau, \lambda^s)$ for all $i \in \mathbb{I}^s$, which is immediate from the Equilibrium Utility condition in (ii): $U^s(i) = \max\{0, \max_{\tau' \in T} \bar{U}^s(i, \tau', \Lambda^s(\tau'))\} \geq \bar{U}^s(i, \tau, \lambda^s)$. Condition 1(b) is $\int U^s dG^s = \int \bar{U}^s dG^s$ at τ, which reduces by separation to the participation constraint $U^s(k^s) = \lambda^s(u^s(k^s, k^{\bar{s}}) - \phi^s)$ from (i). Hence $\tau \in T^p$ with $\Lambda^s(\tau) = \lambda^s$, consistent with (ii).

Step 2: $\bar{V}(\tau') \leq c$ for every $\tau' \in T^p$. Take any $\tau' = (\phi^{s\prime}, G^{s\prime})_{s=a,b} \in T^p$ with $\lambda^{s\prime} = \Lambda^s(\tau')$. By Lemma 1, $(\lambda^{s\prime}, \phi^{s\prime}, G^{s\prime})_{s=a,b}$ satisfies problem (6)'s constraints. Let $k_1^{s\prime} = \max \text{supp}(G^{s\prime})$ on each side.

We construct a separating tuple $(\lambda^{s\prime}, \hat{\phi}^s, \hat{k}^s)_{s=a,b}$ satisfying problem (7)'s constraints with $\hat{\phi}^s \geq \phi^{s\prime}$ for each s; since m is unchanged and $\hat{\phi}^a + \hat{\phi}^b \geq \phi^{a\prime} + \phi^{b\prime}$, this gives $\bar{V}(\hat{\tau}) \geq \bar{V}(\tau')$, and the left-hand side is at most c by (iii), yielding $\bar{V}(\tau') \leq c$. The construction depends on which rates are positive.

Case A: $\lambda^{a\prime}, \lambda^{b\prime} > 0$. The sublattice condition $\omega(k_1^{a\prime}, k_1^{b\prime}) \geq (k_1^{a\prime}, k_1^{b\prime})$ holds by the argument of Part 1's Positive-m case, applied with the weak bound $\Delta^s \geq 0$ (for which non-degeneracy is not needed). By Lemma 10, there is a fixed point $(\hat{k}^a, \hat{k}^b) \geq (k_1^{a\prime}, k_1^{b\prime})$ and an associated $\hat{\tau} \in T^p$ satisfying (7) with fees $\hat{\phi}^s \geq u^s(k_1^{s\prime}, k_1^{\bar{s}\prime}) - U^s(k_1^{s\prime})/\lambda^{s\prime}$. Binding participation at $k_1^{s\prime}$ in τ' gives $U^s(k_1^{s\prime})/\lambda^{s\prime} = \int u^s(k_1^{s\prime}, j) dG^{\bar{s}\prime}(j) - \phi^{s\prime}$, and Common Ranking with $k_1^{\bar{s}\prime} \geq j$ for $j \in \text{supp}(G^{\bar{s}\prime})$ gives $u^s(k_1^{s\prime}, k_1^{\bar{s}\prime}) \geq \int u^s(k_1^{s\prime}, j) dG^{\bar{s}\prime}(j)$. Combining, $\hat{\phi}^s \geq \phi^{s\prime}$.

Case B: $\lambda^{a\prime} > 0 = \lambda^{b\prime}$ (symmetric when sides swapped). Participation in (6) at $\lambda^{b\prime} = 0$

forces $U^b(i) = 0$ on $\text{supp}(G^{b\prime})$, in particular $U^b(k_1^{b\prime}) = 0$. Lemmas 9–10 require both rates positive, so we construct the deviation directly. Set $\hat{k}^b = k_1^{b\prime}$ and let $\hat{k}^a$ be the largest element of $\arg\max_{k' \in \mathbb{I}^a}[u^a(k', \hat{k}^b) - U^a(k')/\lambda^{a\prime}]$ (well-defined by continuity and compactness of $\mathbb{I}^a$). Then set $\hat{\phi}^a = u^a(\hat{k}^a, \hat{k}^b) - U^a(\hat{k}^a)/\lambda^{a\prime}$ and $\hat{\phi}^b = u^b(\hat{k}^b, \hat{k}^a)$. The side-$a$ sublattice argument from Case A (which uses only binding side-a participation at $k_1^{a\prime}$, Supermodularity, and $\lambda^{a\prime} > 0$; no side-b rate enters) gives $\hat{k}^a \geq k_1^{a\prime}$. Problem (7)-feasibility is straightforward: side-a constraints hold by argmax optimality of $\hat{k}^a$ and participation; side-b downward IC and participation are trivial at $\lambda^{b\prime} = 0$ using $U^b(\hat{k}^b) = 0$; rationality holds with equality on side b and from $U^a \geq 0$ on side a. For fee comparisons: on side a, the Case A argument (via argmax optimality and Common Ranking) gives $\hat{\phi}^a \geq \phi^{a\prime}$; on side b, agent rationality at τ' evaluated at $k_1^{b\prime} \in \text{supp}(G^{b\prime})$ gives $\phi^{b\prime} \leq \int u^b(k_1^{b\prime}, j)\, dG^{a\prime}(j) \leq u^b(k_1^{b\prime}, \hat{k}^a) = \hat{\phi}^b$ by Common Ranking (with $\hat{k}^a \geq k_1^{a\prime} \geq j$ for $j \in \text{supp}(G^{a\prime})$).

Case C: $\lambda^{a\prime} = \lambda^{b\prime} = 0$. Participation forces $U^s(k_1^{s\prime}) = 0$ on both sides. Set $\hat{k}^s = k_1^{s\prime}$ and $\hat{\phi}^s = u^s(k_1^{s\prime}, k_1^{\bar{s}\prime})$. Problem (7)'s constraints hold trivially (all IC and participation constraints reduce to $U^s(i) \geq 0$ or $0 = 0$; rationality holds with equality). For fee comparisons: on each side s, agent rationality at τ' evaluated at $k_1^{s\prime}$ gives $\phi^{s\prime} \leq \int u^s(k_1^{s\prime}, j)\, dG^{\bar{s}\prime}(j) \leq u^s(k_1^{s\prime}, k_1^{\bar{s}\prime}) = \hat{\phi}^s$ by Common Ranking. Fee nonnegativity of the deviation follows from $\hat{\phi}^a + \hat{\phi}^b \geq \phi^{a\prime} + \phi^{b\prime} \geq 0$.

Step 3: $\{T^p, \tilde{T}, \Lambda, U\}$ with measures μ, ν^a, ν^b is a CSE. Step 1 gives $T \subseteq T^p$. Hypothesis (i) gives $\bar{V}(\tau) = m(\lambda^a, \lambda^b)(\phi^a + \phi^b) = c$ for every $\tau \in T$, and Step 2 gives $\bar{V}(\tau') \leq c$ for every $\tau' \in T^p$, so every $\tau \in T$ attains $\max_{\tau' \in T^p} \bar{V}(\tau') = c$. Hence $T \subseteq \tilde{T} \subseteq T^p$.

We verify the CSE conditions for $\{T^p, \tilde{T}, \Lambda, U\}$ with measures μ, ν^a, ν^b. *Optimal Search* and *Profit Maximization* hold by construction of T^p, Λ, and $\tilde{T}$. *Free Entry*: $\bar{V}(\tau) = c$ for every $\tau \in \tilde{T}$ by definition of $\tilde{T}$ and Step 2. *Consistency*: holds on T by (ii), and trivially on $\tilde{T} \setminus T$ since μ and ν^s assign zero mass there. *Market Clearing*: by (ii), with ν^s's support $T \subseteq \tilde{T}$, the integral over $\tilde{T}$ equals that over T. *Equilibrium Utility*: (ii) gives $U^s(i) = \max\{0, \max_{\tau \in T} \bar{U}^s\}$; Optimal Search gives $U^s(i) \geq \bar{U}^s(i, \tau', \Lambda^s(\tau'))$ for every $\tau' \in T^p \supseteq \tilde{T}$, so $\max_{\tilde{T}} \bar{U}^s \leq U^s(i)$; and $T \subseteq \tilde{T}$ gives $\max_T \bar{U}^s \leq \max_{\tilde{T}} \bar{U}^s$. Combining, $U^s(i) = \max\{0, \max_{\tilde{T}} \bar{U}^s\}$. ■

Proof of Corollary 1. Fix an active $\tau \in T$ in the full-μ-measure subset satisfying Proposition 1, Part 1. To find a contradiction, suppose there is a violation of slack upward incentive constraints, WLOG on side a: there is $k^s \in \text{supp}(G^s)$, $i \in \mathbb{I}^a$, $U^a(i) = \bar{U}^a(i, \tau, \lambda^a)$, and $u^b(k^b, i) > u^b(k^b, k^a)$. Let $k_1^s = \max(\text{supp}(G^s))$ on each side. By separation, $u^b(k^b, k^a) = u^b(k^b, k_1^a)$, so $u^b(k^b, i) > u^b(k^b, k_1^a)$ and hence $i > k_1^a$ by Common Ranking. Supermodularity (with $k_1^b \geq k^b$) then gives $u^b(k_1^b, i) > u^b(k_1^b, k_1^a)$, so WLOG $k^s = k_1^s$.

Binding upward IC at i gives $U^a(i)/\lambda^a = \int u^a(i,j)\,dG^b(j) - \phi^a$, the same binding-participation property used in Prop 1's proof at k_1^b. Replacing k_1^a with i throughout Part 1's argument, the cream-skimming deviation $\hat{\tau}$ targeting (i, k_1^b) produces $\hat{\phi}^a + \hat{\phi}^b > \phi^a + \phi^b \geq 0$ in both the Positive-m and Zero-m subcases, contradicting Free Entry. ∎

B.3 Monotonicity and Differentiability

Proof of Lemma 2. First assume $U^s(i) > 0$. The fourth condition in the definition of equilibrium utility implies there is a $\tau \in T$ such that $U^s(i) = \bar{U}^s(i, \tau, \Lambda^s(\tau))$. Lemma 1 implies the associated tuple solves problem (6). Eliminate fees from the constraints in problem (6) to get

$$U^s(i') - U^s(i) \geq \lambda^s \int_{\mathbb{I}^{\bar{s}}} (u^s(i', j) - u^s(i, j)) dG^{\bar{s}}(j).$$

With IWTP, the right hand side is positive for all $i' > i$, while with DWTP it is positive for all $i' < i$.

If $U^s(i) = 0$, non-negativity of equilibrium utility implies $U^s(i') \geq U^s(i)$ for all i', and in particular for $i' > i$ with IWTP and $i' < i$ with DWTP. ∎

Proof of Lemma 3. From Lemma 2, Monotone Willingness-to-Pay implies that equilibrium utilities $U^s(i)$ are monotone. By Lebesgue's theorem, any monotone function is differentiable almost everywhere.

Now fix a $\tau \in T$, with $\lambda^s = \Lambda^s(\tau)$ and $k^a \in \text{supp}(G^a)$, $k^a \in (\underline{i}^a, \bar{i}^a)$. Common Ranking and Supermodularity imply $u^a(k^a, j)$ is strictly increasing in j, so Proposition 1, Part 1 implies for a.e. such τ, G^b is degenerate at some k^b.

Since $k^a \in (\underline{i}^a, \bar{i}^a)$, there exists an $\epsilon > 0$ with $\underline{i}^a \leq k^a - \epsilon < k^a + \epsilon \leq \bar{i}^a$. The constraints from problem (6) imply

$$\frac{U^a(k^a + \epsilon) - U^a(k^a)}{\epsilon} \geq \lambda^a \frac{(u^a(k^a + \epsilon, k^b) - u^a(k^a, k^b))}{\epsilon},$$
$$\frac{U^a(k^a) - U^a(k^a - \epsilon)}{\epsilon} \leq \lambda^a \frac{(u^a(k^a, k^b) - u^a(k^a - \epsilon, k^b))}{\epsilon}.$$

Since U^a is differentiable almost everywhere, taking the limit as $\epsilon \to 0$ gives $U^{a\prime}(k^a) = \lambda^a u_1^a(k^a, k^b)$. A symmetric argument establishes the result for side b. ∎

C Appendix for Section 5

C.1 Proofs

Proof of Lemma 4. Suppose, toward contradiction, that $m(\Lambda^a(\tau), \Lambda^b(\tau)) > 0$ on a set of $\tau \in T$ with positive μ-measure. Pick any such τ with fees ϕ^s. By Proposition 1, Part 1, $\lambda^s = \Lambda^s(\tau) > 0$ on each side, τ is separating, and at most one of G^a, G^b is non-degenerate. WLOG $G^b = \delta_{k^b}$ for some $k^b \in \mathbb{I}^b$; pick $k^a \in \text{supp}(G^a)$, taking $k^a = \max \text{supp}(G^a)$ when G^a is non-degenerate. By separation, $u^b(k^b, \cdot)$ is constant on $\text{supp}(G^a)$, so the (P) constraints at τ reduce to point evaluations against k^b. Since $c = 0$ and $m > 0$, free entry gives $\phi^a + \phi^b = 0$, so without loss of generality $\phi^a \leq 0$.

Step 1: A small fee increase attracts a weakly higher type. Consider a perturbed fee $\phi^{a\prime}$ with $\phi^a < \phi^{a\prime} < u^a(k^a, k^b)$. Holding side b fixed, Optimal Search makes the new attracted type the minimizer of $U^a(k')/[u^a(k', k^b) - \phi^{a\prime}]$ over $\mathbb{I}^a$, equivalently the maximizer of

$$G(k', \phi^{a\prime}) \equiv \log\left(u^a(k', k^b) - \phi^{a\prime}\right) - \log U^a(k').$$

The cross-partial of G with respect to $(k', \phi^{a\prime})$ is $u^a_1(k', k^b)/[u^a(k', k^b) - \phi^{a\prime}]^2$, which is non-negative under IWTP. By Topkis's Theorem, the largest-element selection is weakly increasing in $\phi^{a\prime}$, so the new type $k^{a\prime}$ at $\phi^{a\prime} > \phi^a$ satisfies $k^{a\prime} \geq k^a$. For $\phi^{a\prime} - \phi^a$ small, the new contact rate $\lambda^{a\prime}$ satisfies $m(\lambda^{a\prime}, \lambda^b) > 0$ by continuity.

Step 2: If $k^{a\prime} = k^a$. Let δ_k denote the point mass at k. The terms-of-trade $(\phi^{a\prime}, \phi^b, \delta_{k^a}, \delta_{k^b})$ is feasible, induces agent matching probabilities $(\lambda^{a\prime}, \lambda^b)$, and yields profit $m(\lambda^{a\prime}, \lambda^b)(\phi^{a\prime} + \phi^b) > 0 = c$, contradicting optimality.

Step 3: If $k^{a\prime} > k^a$. Apply Lemma 10 at rates $(\lambda^{a\prime}, \lambda^b)$ with starting point $(\underline{k}^a, \underline{k}^b) = (k^{a\prime}, k^b)$. We verify the sublattice condition.

On side a: Step 1 established that $k^{a\prime}$ is the largest maximizer of $u^a(k', k^b) - U^a(k')/\lambda^{a\prime}$, so $\max \Omega^a(k^b) \geq k^{a\prime}$.

On side b: by Optimal Search for the original τ, k^b maximizes $u^b(k', k^a) - U^b(k')/\lambda^b$ over $\mathbb{I}^b$. For any $\tilde{k} < k^b$ and any partner $\tilde{k}^a \geq k^{a\prime} > k^a$, Common Ranking and Supermodularity give

$$u^b(k^b, \tilde{k}^a) - u^b(\tilde{k}, \tilde{k}^a) \geq u^b(k^b, k^a) - u^b(\tilde{k}, k^a) \geq \frac{U^b(k^b) - U^b(\tilde{k})}{\lambda^b},$$

where the first inequality uses Supermodularity (with $k^b > \tilde{k}$ and $\tilde{k}^a \geq k^a$) and the second uses the b-side incentive constraint at the original τ. Rearranging, k^b weakly dominates $\tilde{k}$ in

$\Omega^b(\tilde{k}^a)$, so $\max \Omega^b(k^{a\prime}) \geq k^b$.

The lemma yields $\hat{\tau} \in T^p$ with fees $\hat{\phi}^a \geq u^a(k^{a\prime}, k^b) - U^a(k^{a\prime})/\lambda^{a\prime} = \phi^{a\prime}$ (the last equality by definition of $k^{a\prime}$ as the attracted type at fee $\phi^{a\prime}$) and $\hat{\phi}^b \geq u^b(k^b, k^{a\prime}) - U^b(k^b)/\lambda^b \geq u^b(k^b, k^a) - U^b(k^b)/\lambda^b = \phi^b$ (using Common Ranking with $k^{a\prime} > k^a$ and Optimal Search at τ). Since $\phi^{a\prime} + \phi^b > \phi^a + \phi^b = 0$, we have $\hat{\phi}^a + \hat{\phi}^b > 0$, so $\bar{V}(\hat{\tau}) = m(\lambda^{a\prime}, \lambda^b)(\hat{\phi}^a + \hat{\phi}^b) > 0 = c$, contradicting Free Entry. ■

Verification for CES payoffs. We now show that the CES specification satisfies Assumption 7 when $\theta < 1 + \gamma$ and $\underline{i} = 0$. Assume CES payoff and matching functions:

$$u(i,j) = \left(\tfrac{1}{2} i^{\frac{\theta-1}{\theta}} + \tfrac{1}{2} j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \text{ and } m(\lambda^a, \lambda^b) = \left(1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma\right)^{\frac{1}{\gamma}}.$$

This implies $\mathbb{\Lambda} = \{(\lambda^a, \lambda^b) | (\lambda^a)^\gamma + (\lambda^b)^\gamma \leq 1\}$, and $\bar{\lambda} = 2^{-1/\gamma}$. This implies $u(i,i) = i$ and $u_1(i,i) = u_2(i,i) = \frac{1}{2}$ for all i and so $U(i) = \bar{\lambda} i/2$ and $\Phi(i) = i/2$ for all i.

For condition (a), the condition $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}\right) \notin \mathbb{\Lambda}$ can be expressed as

$$\tfrac{1}{2} i^{\frac{\gamma}{\theta}} + \tfrac{1}{2} j^{\frac{\gamma}{\theta}} < \left(\tfrac{1}{2} i^{\frac{\theta-1}{\theta}} + \tfrac{1}{2} j^{\frac{\theta-1}{\theta}}\right)^{\frac{\gamma}{\theta-1}} = \left(\tfrac{1}{2}\left(i^{\frac{\gamma}{\theta}}\right)^{\frac{\theta-1}{\gamma}} + \tfrac{1}{2}\left(j^{\frac{\gamma}{\theta}}\right)^{\frac{\theta-1}{\gamma}}\right)^{\frac{\gamma}{\theta-1}}.$$

Using Jensen's inequality, this holds for all $i \neq j$ if and only if $\gamma > \theta - 1$. Conversely, if $\theta > 1 + \gamma$, there is no PAM equilibrium.

For condition (b), consider a $(\underline{i}, j)$ market with $\underline{i} = 0$. If $\theta \leq 1$, $u_1(j, 0) = 0$ for all $j > 0$, so the $(j, 0)$ market is infeasible as shown above. For $\theta > 1$,

$$\tilde{\ell}(j) = \bar{\lambda}\frac{u_1(j,j)}{u_1(j,0)} = 2^{\frac{1+\gamma-\theta}{\gamma(\theta-1)}}.$$

This exceeds 1, meaning the market is infeasible, whenever $\theta < 1 + \gamma$. This proves that for $\theta < 1 + \gamma$ and $\underline{i} = 0$, the CES specification satisfies Assumption 7, and so a PAM CSE exists. For larger values of θ, such an equilibrium cannot exist.

Proof of Proposition 2. In a symmetric PAM CSE, type i matches with type i in every active market. By Lemma 4, $m(\lambda, \lambda) = 0$ at μ-a.e. active terms-of-trade, so $\lambda = \bar{\lambda}$ in a symmetric CSE. Lemma 3 then gives $U'(i) = \bar{\lambda} u_1(i,i)$ at all interior types where U is differentiable. To pin down the level, note that the lowest type $\underline{i}$ faces no downward incentive constraints (there are no lower types to exclude). The $(\underline{i}, \underline{i})$ market therefore solves the unconstrained problem

$$\max_{\lambda\in[0,\bar{\lambda}]} 2m(\lambda,\lambda)\left(u(\underline{i},\underline{i})-\frac{U(\underline{i})}{\lambda}\right),$$

which, given $\lambda=\bar{\lambda}$ and $m(\bar{\lambda},\bar{\lambda})=0$, requires $U(\underline{i})=\bar{\lambda}u(\underline{i},\underline{i})$. Together these determine $U(i)=\bar{\lambda}u(\underline{i},\underline{i})+\bar{\lambda}\int_{\underline{i}}^{i}u_1(x,x)\,dx$ for all i. The participation constraint gives fees $\Phi(i)=u(i,i)-U(i)/\bar{\lambda}$, which implies $\Phi(\underline{i})=0$ and $\Phi'(i)=u_2(i,i)>0$.

Verifying Downward ICs. Using the characterization of equilibrium utility, for $j<i$ we have

$$U(j)-U(i)=-\bar{\lambda}\int_j^i u_1(x,x)dx>-\bar{\lambda}\int_j^i u_1(x,i)dx=\bar{\lambda}(u(j,i)-u(i,i)),$$

where in the first equality we use $U'(i)=\bar{\lambda}u_1(i,i)$, in the inequality we use Supermodularity, and in the last equality we solve out the integral. This is the global downward incentive constraint.

Existence of a PAM CSE. We still need to rule out the possibility that an (i,j) market is more profitable for $i\neq j$. First, assume that i and j are both strictly above $\underline{i}$, so in a proposed (i,j) market, we have a binding local incentive constraint on both sides:

$$U'(i)=\lambda^a u_1(i,j),\quad U'(j)=\lambda^b u_1(j,i).$$

Using $U'(i)=\bar{\lambda}u_1(i,i)$ and $U'(j)=\bar{\lambda}u_1(j,j)$, we can write this as

$$\lambda^a=\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)},\quad \lambda^b=\bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}.$$

By Assumption 7(a), however, $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)},\bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}\right)\notin\Lambda^o$, so any such terms-of-trade is either infeasible or delivers zero platform matching probability.

Now consider a proposed $(i,\underline{i})$ market with $i>\underline{i}$, where side a attracts type i and side b attracts type $\underline{i}$. Type $\underline{i}$ is the lowest type and faces no downward incentive constraints, so the analysis differs from the previous case.

On side a, the downward incentive constraints require $U(k)-U(i)\geq\lambda^a(u(k,\underline{i})-u(i,\underline{i}))$ for all $k<i$. If $u_1(k,\underline{i})=0$ for all k, then for any $k<i$

$$U(k)-U(i)=-\bar{\lambda}\int_k^i u_1(x,x)\,dx<-\bar{\lambda}\int_k^i u_1(x,\underline{i})\,dx=0,$$

where the inequality uses supermodularity and the last equality uses $u_1(\cdot, \underline{i}) = 0$. Since $\lambda^a(u(k, \underline{i}) - u(i, \underline{i})) = 0$ as well, the downward IC is violated for any λ^a, so the $(i, \underline{i})$ market is infeasible. This is the first case in Assumption 7(b).

If $u_1(i, \underline{i}) > 0$, the local IC on side a pins down $\lambda^a = \tilde{\ell}(i) \equiv \bar{\lambda}\, u_1(i,i)/u_1(i, \underline{i})$. The participation constraints give fees $\phi^a = u(i, \underline{i}) - U(i)/\lambda^a = \tilde{\phi}(i)$ and $\phi^b = u(\underline{i}, i) - U(\underline{i})/\lambda^b = u(\underline{i}, i) - \bar{\lambda}\, u(\underline{i}, \underline{i})/\lambda^b$. Their sum is

$$\phi^a + \phi^b = u(\underline{i}, i) + \tilde{\phi}(i) - \frac{\bar{\lambda}\, u(\underline{i}, \underline{i})}{\lambda^b}.$$

If $u(\underline{i}, i) + \tilde{\phi}(i) \leq 0$, then $\phi^a + \phi^b \leq 0$ for all $\lambda^b > 0$, so the market cannot generate positive fees. This is the second case in Assumption 7(b).

If $u(\underline{i}, i) + \tilde{\phi}(i) > 0$, positive fees require $\lambda^b > \bar{\lambda}\, u(\underline{i}, \underline{i})/(u(\underline{i}, i) + \tilde{\phi}(i))$. For the market to be both feasible and profitable, we need $(\tilde{\ell}(i), \lambda^b) \in \mathbb{\Lambda}^o$ for some such λ^b. Assumption 7(b) rules this out by requiring $\big(\tilde{\ell}(i),\, \bar{\lambda}\, u(\underline{i}, \underline{i})/(u(\underline{i}, i) + \tilde{\phi}(i))\big) \notin \mathbb{\Lambda}^o$. Since m is strictly decreasing in its second argument, any larger λ^b also has $(\tilde{\ell}(i), \lambda^b) \notin \mathbb{\Lambda}^o$, making the market unprofitable or infeasible. This is the third case in Assumption 7(b).

The remaining CSE conditions follow from Proposition 1, Part 2: take $\nu^a = \nu^b$ to be the pushforward of $I\, dF$ on $\mathbb{I}$ under the map $i \mapsto \tau(i)$, set $d\mu = \bar{\lambda}\, d\nu^s$, and verify Consistency (by construction, since $\Lambda^s(\tau(i)) = \bar{\lambda}$), Market Clearing (with equality by construction), and Equilibrium Utility (since $U(i) = \bar{\lambda}(u(i,i) - \Phi(i))$ is achieved at $\tau(i)$). ■

C.2 Anchor-Matching Equilibrium

We exhibit an anchor-matching CSE for the parametric specification of Section 5: $u(i,j) = \sqrt{ij}$, $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ (so $\gamma = 1$ and $\bar{\lambda} = 1/2$), with types in $[\underline{i}, \bar{i}]$ where $0 < \underline{i} < \bar{i}$.

Under this specification, Assumption 7(a) holds: for any $i > j > \underline{i}$, the matching probabilities required by (a) are $\big(\frac{1}{2}\sqrt{i/j},\, \frac{1}{2}\sqrt{j/i}\big)$, whose sum $\frac{1}{2}\big(\sqrt{i/j} + \sqrt{j/i}\big) > 1$ by AM–GM, putting the pair outside $\mathbb{\Lambda}$. On the other hand, Assumption 7(b) is violated: at the figure's parameters $\underline{i} = 1$, $\bar{i} = 2$ and any i sufficiently close to $\underline{i}$, all three conditions of (b) fail. The first fails since $u_1(i, 1) = \frac{1}{2\sqrt{i}} > 0$. Direct computation gives $\tilde{\phi}(i) = \frac{i-1}{2\sqrt{i}}$, so $u(\underline{i}, i) + \tilde{\phi}(i) = \frac{3i-1}{2\sqrt{i}} > 0$, ruling out the second. And $\tilde{\ell}(i) + \bar{\lambda}\frac{u(\underline{i},\underline{i})}{u(\underline{i},i)+\tilde{\phi}(i)} = \frac{\sqrt{i}(1+3i)}{6i-2} \approx \frac{5-i}{4} < 1$, where the approximation is the first-order expansion at $i = 1$. This means that $\left(\tilde{\ell}(i), \bar{\lambda}\frac{u(\underline{i},\underline{i})}{u(\underline{i},i)+\tilde{\phi}(i)}\right) \in \mathbb{\Lambda}^o$ for all i near $\underline{i}$.

Since Assumption 7(b) is violated, Proposition 2 does not apply. Instead, we look for a CSE with the following characterization: there is an anchor type $i^* \in (\underline{i}, \bar{i})$ such that:

1. there is an undistorted terms-of-trade matching i^* with i^*, delivering equilibrium utility $U(i^*) = \bar{\lambda} u(i^*, i^*)$;

2. for all type $i > i^*$, there is a terms-of-trade matching i with i with matching probability $\bar{\lambda}$, and binding local incentive constraints, so $U'(i) = \bar{\lambda} u_1(i,i)$;

3. all types $i < i^*$ match with some type $\sigma(i) > i^*$, where σ is strictly decreasing, so types in $(i^*, \sigma(\underline{i})]$ each participate in two terms-of-trade: one with their own type and one as the anchor partner for some $i < i^*$.

This third matching pattern is novel. The remainder of this section characterizes such an anchor CSE and shows that one exists in our example. First we find equations describing who matches with whom ($\sigma(i)$) and equilibrium utility ($U(i)$) for a given i^*; then we show how to determine the anchor i^*; finally, we verify global incentive constraints, prove there are no profitable platform deviations, and show that market clearing is satisfied.

Characterizing Equilibrium: σ and U. Fix a candidate anchor i^*. Above the anchor, the equilibrium replicates a PAM CSE whose lowest type is i^*: the local incentive constraint with self-matching at probability $\bar{\lambda}$ gives

$$U'(i) = \bar{\lambda}\, u_1(i,i), \tag{19}$$

with boundary $U(i^*) = \bar{\lambda}\, u(i^*, i^*)$.

Below the anchor, let $(\ell^a(i), \ell^b(i))$ denote the matching probabilities at an $(i, \sigma(i))$ terms-of-trade and $(\Phi^a(i), \Phi^b(i))$ denote the fees. Then if an $(i, \sigma(i))$ terms-of-trade is profit-maximizing, we have both $m(\ell^a(i), \ell^b(i)) = 0$ and $\Phi^a(i) + \Phi^b(i) = 0$. Free entry implies that one or the other must be zero. We find that *both* must vanish to block deviations (profitable creation of alternative terms-of-trade) in two directions. Markets matching i to $j > \sigma(i)$ are infeasible: the local-IC matching probabilities at a more-distinct pair lie outside $\mathbb{\Lambda}$ under the complementarity captured by Assumption 7(a). This pushes equilibrium markets to the boundary of $\mathbb{\Lambda}$, where $m = 0$. Markets matching i to $j < \sigma(i)$ are feasible ($m > 0$) and so are blocked only if the sum of fees is nonpositive; by continuity, the sum of fees is zero at $(i, \sigma(i))$. We verify both these claims when we discuss profitable deviations below.

In terms of U and σ (writing the contact rates via the local ICs), the two conditions read

$$m\left(\frac{U'(i)}{u_1(i,\sigma(i))}, \frac{U'(\sigma(i))}{u_1(\sigma(i),i)}\right) = 0, \tag{20}$$

$$u(i,\sigma(i)) + u(\sigma(i),i) - \frac{U(i)}{U'(i)} u_1(i,\sigma(i)) - \frac{U(\sigma(i))}{U'(\sigma(i))} u_1(\sigma(i),i) = 0. \tag{21}$$

Since $\sigma(i) > i^*$, the values $U(\sigma(i))$ and $U'(\sigma(i))$ are fixed by (19), so (20)–(21) form a coupled system for $U(i)$ and $\sigma(i)$ on the anchor segment with initial conditions $\sigma(i^*) = i^*$ and $U(i^*) = \bar{\lambda} u(i^*, i^*)$. These equations apply to any utility with IWTP.

Specializing to $u(i,j) = \sqrt{ij}$ and $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$, the PAM segment gives $U(i) = (i + i^*)/4$ for $i \geq i^*$, hence $U(\sigma(i)) = (\sigma(i) + i^*)/4$ and $U'(\sigma(i)) = 1/4$. Equation (20) becomes $\ell^a(i) + \ell^b(i) = 1$ with $\ell^b(i) = \frac{1}{2}\sqrt{\sigma(i)/i}$, so

$$U'(i) = \tfrac{1}{2}\sqrt{\sigma(i)/i}\,\bigl(1 - \tfrac{1}{2}\sqrt{\sigma(i)/i}\bigr). \tag{22}$$

Note that $U'(i) \leq 1/4$ globally, since $y(1-y) \leq 1/4$. Equation (21) yields $U(i)/U'(i) = i(3\sigma(i) - i^*)/\sigma(i)$. Eliminating U' using equation (22) gives

$$U(i) = \frac{(2\sqrt{i} - \sqrt{\sigma(i)})(3\sigma(i) - i^*)}{4\sqrt{\sigma(i)}}.$$

Differentiating and equating to (22),

$$\sigma'(i) = -\frac{\sigma(i)\bigl(\sigma(i)^{3/2} + \sqrt{i}\,(\sigma(i) - i^*)\bigr)}{i\bigl(\sqrt{i}\,i^* - 3\sigma(i)(\sqrt{\sigma(i)} - \sqrt{i})\bigr)}. \tag{23}$$

At the anchor, $\sigma(i^*) = i^*$, so $\sigma'(i^*) = -1$. More generally, this equation holds as long as the denominator is positive, yielding $\sigma'(i) < 0$: negative assortative matching along the anchor segment. We further discuss non-negativity of the denominator below.

Pinning Down the Anchor. The anchor i^* is determined by competition for the lowest type $\underline{i}$, who alone faces no downward incentive constraint and so is uniquely attractive. If i^* is too small, $U(\underline{i})$ is too low to compete away profits: a platform can profitably create a terms-of-trade matching $\underline{i}$ to some $j > i^*$, breaking the candidate. Raising i^* boosts the entire utility profile and shrinks the profit margin on such deviations. At a critical i^*, the most profitable $(\underline{i}, j)$ deviation just breaks even and no profitable deviation remains. Raising i^* beyond this point would push utility so high that there is no $(\underline{i}, j)$ terms-of-trade with nonnegative fees and feasible matching probabilities. This means the lowest types would have nowhere to be matched, violating market clearing. Thus we pin down a unique i^*.

To formalize, consider a candidate market matching $\underline{i}$ on side a to some $j > \underline{i}$ on side b. The side-b local IC fixes $\bar{\lambda}^b(j) = U'(j)/u_1(j, \underline{i})$. Then if $(0, \bar{\lambda}^b(j)) \notin \Lambda$, the market is infeasible. Otherwise let $\bar{\lambda}^a(j) \geq 0$ solve $m(\bar{\lambda}^a(j), \bar{\lambda}^b(j)) = 0$. This is the largest side-a probability consistent with feasibility. Define

$$H(j) \equiv \bar{\lambda}^a(j)\left(u(\underline{i}, j) + u(j, \underline{i}) - \frac{U(j)}{U'(j)}\,u_1(j, \underline{i})\right).$$

The $(\underline{i}, j)$ market is strictly profitable iff $H(j) > U(\underline{i})$: shrinking λ^a slightly below $\bar{\lambda}^a(j)$ delivers a market with strictly positive matching probability and strictly positive sum of fees. Equilibrium therefore requires $\max_j H(j) = U(\underline{i})$.

By construction of σ, the zero-fee condition (21) at $i = \underline{i}$ gives exactly $H(\sigma(\underline{i})) = U(\underline{i})$. So at the critical i^*, $\sigma(\underline{i}) \in \arg\max_j H(j)$. Two configurations satisfy this: either $\sigma(\underline{i}) = \bar{i}$, with all types in the anchor segment; or $\sigma(\underline{i}) \in (\underline{i}, \bar{i})$ is an interior stationary point of H. In the latter case, $H'(\sigma(\underline{i})) = 0$ makes $(\underline{i}, \sigma(\underline{i}))$ a singular point of (20)–(21), at which σ' diverges.

To see why, define $\hat{U}(i, j) = \bar{U}^s(i, \tau, \Lambda^s(\tau))$ at a terms-of-trade τ matching (i, j). Then $U(i) = \hat{U}(i, \sigma(i))$ for $i \leq i^*$. Differentiating with respect to i and solving for $\sigma'(i)$,

$$\sigma'(i) = \frac{U'(i) - \hat{U}_1(i, \sigma(i))}{\hat{U}_2(i, \sigma(i))}.$$

Now $H(j) = \hat{U}(\underline{i}, j)$ identically in j: both are the equilibrium utility delivered to type $\underline{i}$ by a market matching $\underline{i}$ with j at feasible matching probabilities and zero sum of fees. Differentiating, $\hat{U}_2(\underline{i}, \sigma(\underline{i})) = H'(\sigma(\underline{i}))$, and so

$$\sigma'(\underline{i}) = \frac{U'(\underline{i}) - \hat{U}_1(\underline{i}, \sigma(\underline{i}))}{H'(\sigma(\underline{i}))}.$$

When $\sigma(\underline{i})$ is an interior maximizer of H, the denominator vanishes and $\sigma'(\underline{i})$ diverges. In our parametric case, this condition reduces to the denominator of equation (23) vanishing at $i = \underline{i}$:

$$i^* = 3\sigma(\underline{i}) \left(\sqrt{\frac{\sigma(\underline{i})}{\underline{i}}} - 1 \right).$$

This delivers a practical approach for numerical computation: fix i^* and solve (23) downward from i^*. Then adjust i^* until either $\sigma'(\underline{i}) \to -\infty$ or $\sigma(\underline{i}) = \bar{i}$.

Downward Incentive Constraints. For $i \geq i^*$, $U'(i) = 1/4$. For $i < i^*$, $U'(i)$ satisfies equation (22). Since $\sigma(i) > i$, $\frac{1}{2}\sqrt{\sigma(i)/i}\left(1 - \frac{1}{2}\sqrt{\sigma(i)/i}\right)$ is decreasing in $\sigma(i)/i$, hence increasing in i, with $U'(i^*) = 1/4$. This proves U is globally weakly convex.

Define $V(t) \equiv U(t^2)$ so $V'(t) = 2tU'(t^2)$ and $V''(t) = 2U'(t^2) + 4t^2U''(t^2)$. Since U is strictly increasing and weakly convex, V is strictly convex.

Next, consider the incentive constraint keeping type k out of the side of any equilibrium market where the designated type is i and the partner type is j:

$$U(k) - U(i) \geq \lambda^s(u(k,j) - u(i,j)) = \lambda^s \sqrt{j}\,(\sqrt{k} - \sqrt{i}).$$

Using the local incentive constraint $\lambda^s = U'(i)/u_1(i,j) = 2\sqrt{i}\,U'(i)/\sqrt{j}$ and the definition of V, this reduces to

$$V(\sqrt{k}) - V(\sqrt{i}) \geq V'(\sqrt{i})(\sqrt{k} - \sqrt{i}).$$

Strict convexity of V implies this holds as a strict inequality.

No Profitable Deviations. We now show there are no profitable deviations, dividing the possible deviations (i, j) into four cases: (1) $i, j \geq i^*$; (2) $\underline{i} < i < i^*$ and $j \geq \sigma(i)$; (3) $\underline{i} < i < i^*$ and $i^* \leq j < \sigma(i)$; and (4) $\underline{i} < i, j < i^*$.

Start with the first case: the local ICs require $\lambda^a = \frac{1}{2}\sqrt{i/j}$ and $\lambda^b = \frac{1}{2}\sqrt{j/i}$, so $\lambda^a + \lambda^b = \frac{1}{2}(\sqrt{i/j} + \sqrt{j/i}) > 1$ for all $i \neq j$ (by AM-GM), making the market infeasible.

In the second case, define $\mu(i,j) \equiv 1 - \frac{U'(i)}{u_1(i,j)} - \frac{U'(j)}{u_1(j,i)}$, the platform matching probability in a proposed (i, j) market. Because $j > i^*$, the positive assortative matching segment gives $U'(j) = \frac{1}{4}$. Differentiating and using the functional form for u,

$$\mu_2(i,j) = \frac{U'(i)\sqrt{i}}{j^{3/2}} - \frac{1}{4\sqrt{ij}}.$$

Recalling from equation (22) and the surrounding discussion that $U'(i) \leq \frac{1}{4}$, we have

$$\mu_2(i,j) \leq \frac{i - j}{4j^{3/2}\sqrt{i}} < 0$$

for $i < j$. So $\mu(i, j)$ is decreasing in j. Since $\mu(i, \sigma(i)) = 0$, any (i, j) with $j > \sigma(i)$ leads to matching probabilities that are infeasible.

In the third case, $U(j) = U(i^*) + \frac{1}{4}(j - i^*)$. The sum of fees in a proposed (i, j) market is

$$\Psi(i,j) = 2\sqrt{ij} - \frac{i(3\sigma - i^*)}{\sigma} \cdot \frac{\sqrt{j}}{2\sqrt{i}} - (i^* + j) \cdot \frac{\sqrt{i}}{2\sqrt{j}},$$

where we use $\frac{U(i)}{U'(i)} = \frac{i(3\sigma - i^*)}{\sigma}$ from equations (20)–(21) and $\frac{U(j)}{U'(j)} = i^* + j$ from the positive assortative matching segment, with $\sigma = \sigma(i)$. Differentiating,

$$\Psi_2(i,j) = \frac{\sqrt{i}\,i^*(j + \sigma)}{4j^{3/2}\sigma} > 0.$$

Since $\Psi(i, \sigma(i)) = 0$ and $\Psi_2 > 0$, the sum of fees is negative for any $i^* \leq j < \sigma(i)$. This rules out all deviations in the third case.

In the last case, $i, j < i^*$, the sum of fees is $2\sqrt{ij} - \frac{U(i)}{U'(i)} \cdot \frac{\sqrt{j}}{2\sqrt{i}} - \frac{U(j)}{U'(j)} \cdot \frac{\sqrt{i}}{2\sqrt{j}}$. Since $U'(j) \leq \frac{1}{4}$ and $U(j) \geq U(i^*) + \frac{1}{4}(j - i^*) = \frac{1}{4}(i^* + j)$, we have $\frac{U(j)}{U'(j)} \geq i^* + j$. Thus the sum of fees is bounded above by $\Psi(i, j)$, which is negative for $j < \sigma(i)$. This rules out all profitable deviations.

Market Clearing. For type $i < i^*$, side-a agents flow only through the negative assortative market $(i, \sigma(i))$. Type $\sigma(i) > i^*$, however, appears on side b in two distinct markets: the negatively-matched market with side-a partner i, and the positive assortative market with side-b partner $\sigma(i)$. Decompose the side-b agent flow at type $\sigma(i)$ as

$$d\nu^b\big|_{\text{type } \sigma(i)} = d\nu^b_{\text{NAM}} + d\nu^b_{\text{PAM}},$$

where the first piece carries side-b type-$\sigma(i)$ agents into the $(i, \sigma(i))$ market and the second carries them into the $(\sigma(i), \sigma(i))$ market. Consistency at the NAM market gives $d\mu_{\text{NAM}} = \ell^a(i)\, d\nu^a = \ell^b(i)\, d\nu^b_{\text{NAM}}$, so $d\nu^b_{\text{NAM}}/d\nu^a = \ell^a(i)/\ell^b(i)$. Pushing forward through the inverse map $i \mapsto \sigma(i)$, side-b type-$\sigma(i)$ flow into the NAM market per unit type-$\sigma(i)$ population is $\big(\ell^a(i)/\ell^b(i)\big)\big|di/d\sigma\big| = \big(\ell^a(i)/\ell^b(i)\big)/(-\sigma'(i))$, using $\sigma' < 0$.

Feasibility dictates that this fraction must lie in $[0, 1]$, with the rest absorbed by the PAM market:

$$\frac{\ell^a(i)}{\ell^b(i)} \leq -\sigma'(i).$$

Since $\sigma(i) > i$, we have $\ell^b(i) = \frac{1}{2}\sqrt{\sigma(i)/i} > \frac{1}{2}$ and $\ell^a(i) < \frac{1}{2}$, so $\ell^a(i)/\ell^b(i) < 1$. From equation (23), since $\sigma(i) > i^* > i$, the numerator exceeds $\sigma(i)^{5/2}$ and the denominator (which must be positive for a valid solution) is less than $i^{3/2} i^*$, which in turn is smaller than $i^{3/2}\sigma(i)$. This proves $-\sigma'(i) > (\sigma(i)/i)^{3/2}$. Then $\sigma(i) > i$ gives $-\sigma'(i) > 1$ and hence $-\sigma'(i) > \ell^a(i)/\ell^b(i)$.

D Appendix for Section 6

D.1 Proofs

Proof of Lemma 5. Suppose, toward contradiction, that $\phi^a + \phi^b > 0$ on a set of $\tau \in T$ with positive μ-measure. Pick any such τ with fees ϕ^s. By Proposition 1, Part 1, $\lambda^s = \Lambda^s(\tau) > 0$ on each side, τ is separating, and at most one of G^a, G^b is non-degenerate. WLOG $G^b = \delta_{k^b}$ for some $k^b \in \mathbb{I}^b$; pick $k^a \in \operatorname{supp}(G^a)$, taking $k^a = \max \operatorname{supp}(G^a)$ when G^a is non-degenerate.

By separation, $u^b(k^b, \cdot)$ is constant on $\text{supp}(G^a)$, so the (P) constraints at τ reduce to point evaluations against k^b. Since $c = 0$ and $\phi^a + \phi^b > 0$, free entry gives $m(\lambda^a, \lambda^b) = 0$.

Step 1: A small fee decrease attracts a weakly higher type. Consider a perturbed fee $\phi^{a\prime}$ with $-\phi^b < \phi^{a\prime} < \phi^a$, so $\phi^{a\prime} + \phi^b > 0$ is preserved. Holding side b fixed, Optimal Search makes the new attracted type the minimizer of $U^a(k')/[u^a(k', k^b) - \phi^{a\prime}]$ over $\mathbb{I}^a$, equivalently the maximizer of

$$G(k', \phi^{a\prime}) \equiv \log\left(u^a(k', k^b) - \phi^{a\prime}\right) - \log U^a(k').$$

The cross-partial of G with respect to $(k', \phi^{a\prime})$ is $u_1^a(k', k^b)/[u^a(k', k^b) - \phi^{a\prime}]^2$, which is non-positive under DWTP (where $u^a(\cdot, k^b)$ is weakly decreasing). By Topkis's Theorem, the largest-element selection is weakly decreasing in $\phi^{a\prime}$, so the new type $k^{a\prime}$ at $\phi^{a\prime} < \phi^a$ satisfies $k^{a\prime} \geq k^a$. Since $\phi^{a\prime} < \phi^a$, the new contact rate $\lambda^{a\prime} = U^a(k^{a\prime})/[u^a(k^{a\prime}, k^b) - \phi^{a\prime}]$ is strictly below λ^a. Since $\mathbb{\Lambda}$ is a down set (Section 2.2), $(\lambda^{a\prime}, \lambda^b) \in \mathbb{\Lambda}$; and since m is strictly decreasing in its first argument, $m(\lambda^{a\prime}, \lambda^b) > m(\lambda^a, \lambda^b) = 0$.

Step 2: If $k^{a\prime} = k^a$. Let δ_k denote the point mass at k. The terms-of-trade $(\phi^{a\prime}, \phi^b, \delta_{k^a}, \delta_{k^b})$ is feasible, induces agent matching probabilities $(\lambda^{a\prime}, \lambda^b)$, and yields profit $m(\lambda^{a\prime}, \lambda^b)(\phi^{a\prime} + \phi^b) > 0 = c$, contradicting optimality.

Step 3: If $k^{a\prime} > k^a$. Apply Lemma 10 at rates $(\lambda^{a\prime}, \lambda^b)$ with starting point $(\underline{k}^a, \underline{k}^b) = (k^{a\prime}, k^b)$. The sublattice condition is verified exactly as in Step 3 of the proof of Lemma 4: on side a, $k^{a\prime}$ is the largest maximizer of the a-side objective at $\phi^{a\prime}$ by Step 1 above; on side b, k^b weakly dominates any $\tilde{k} < k^b$ against any partner $\tilde{k}^a \geq k^{a\prime} > k^a$ by Common Ranking, Supermodularity, and the b-side incentive constraint at the original τ. The lemma yields $\hat{\tau} \in T^p$ with fees $\hat{\phi}^a \geq u^a(k^{a\prime}, k^b) - U^a(k^{a\prime})/\lambda^{a\prime} = \phi^{a\prime}$ (the equality by definition of $k^{a\prime}$) and $\hat{\phi}^b \geq u^b(k^b, k^{a\prime}) - U^b(k^b)/\lambda^b \geq u^b(k^b, k^a) - U^b(k^b)/\lambda^b = \phi^b$ (using Common Ranking with $k^{a\prime} > k^a$ and Optimal Search at τ). Since $\phi^{a\prime} + \phi^b > 0$ by choice of $\phi^{a\prime}$, we have $\hat{\phi}^a + \hat{\phi}^b > 0$, so $\bar{V}(\hat{\tau}) = m(\lambda^{a\prime}, \lambda^b)(\hat{\phi}^a + \hat{\phi}^b) > 0 = c$, contradicting Free Entry. ■

Verification for the disease model. We now show that the disease model specification $u(i, j) = 1 - \kappa\, i(1 - j)$ satisfies Assumption 8 when $\underline{i} = 0$. Verifying condition (a) of Assumption 8 is straight algebra using the functional form for u: the inequality simplifies to $(i - j)^2/((1 - i)(1 - j)) \geq 0$, which holds for all i and j.

For condition (b), we rule out $(0, i)$ markets with $i > 0$. The restriction $\underline{i} = 0$ is needed here: since $u(0, j) = 1$ for all j, the lowest type is indifferent about partner quality, which limits the profitability of $(0, i)$ deviations. When $\underline{i} > 0$, $u(\underline{i}, i) > u(\underline{i}, \underline{i})$ for $i > \underline{i}$, so the platform can extract surplus from the lowest type's preference for better partners, and the feasibility conditions in Assumption 8(b) may fail.

Solving the differential equations characterizing the PAM equilibrium with initial condition $\ell(0) = \bar{\lambda}$, we obtain the equilibrium utility $U(i) = 2^{-\frac{1}{\gamma}} e^{-\zeta(i)} \sqrt{1 - \kappa i(1-i)}$, where

$$\zeta(i) \equiv \sqrt{\frac{\kappa}{4-\kappa}} \left(\arctan\left(\sqrt{\frac{\kappa}{4-\kappa}} \right) - \arctan\left(\sqrt{\frac{\kappa}{4-\kappa}} (1-2i) \right) \right).$$

Under a PAM equilibrium, the local IC on side b of this market requires that

$$\tilde{\ell}(i) = \frac{U'(i)}{u_1(i,0)} = 2^{-\frac{1}{\gamma}} \frac{1-i}{\sqrt{1-\kappa i(1-i)}} e^{-\zeta(i)}.$$

The zero-fee condition and the participation constraint on side a then pin down λ^a:

$$\lambda^a = 2^{-\frac{1}{\gamma}} \frac{1-i}{1-2i}.$$

For λ^a to be a positive number, it has to be $i \in \left[0, \frac{1}{2}\right)$. We want to find conditions such that $\Xi(i) \equiv 1 - (\lambda^a)^\gamma - (\tilde{\ell}(i))^\gamma \leq 0$, for $i \in \left[0, \frac{1}{2}\right)$. From the definition, $\Xi(0) = 0$. When $\kappa \leq \min\{2(1+\gamma), 4\}$, $\Xi'(i) \leq 0$ for all $i \in \left[0, \frac{1}{2}\right)$. Since $\Xi(0) = 0$, this proves that condition (b) of Assumption 8 is met when $\kappa \leq \min\{2(1+\gamma), 4\}$.

Proof of Proposition 3. We follow the structure of the proof of Proposition 2.

Characterization. In a symmetric PAM CSE, type i matches with type i in every active market. By Lemma 5, $\Phi(i) = 0$ at μ-a.e. active terms-of-trade. Zero fees imply $U(i) = \ell(i)u(i,i)$. Totally differentiating this condition, we have $U'(i) = \ell'(i)u(i,i) + \ell(i)(u_1(i,i) + u_2(i,i))$. Imposing the local IC $U'(i) = \ell(i)u_1(i,i)$ (Lemma 3), we have $\ell'(i)u(i,i) = -\ell(i)u_2(i,i)$. We can then divide the condition $U'(i) = \ell(i)u_1(i,i)$ by $U(i) = \ell(i)u(i,i)$ to get $\frac{U'(i)}{U(i)} = \frac{u_1(i,i)}{u(i,i)}$. To pin down the level of U and ℓ, we note again that the $(\underline{i}, \underline{i})$ market solves the unconstrained problem, which gives $\ell(\underline{i}) = \bar{\lambda}$ and $U(\underline{i}) = \bar{\lambda} u(\underline{i}, \underline{i})$.

Verifying Downward ICs. Take $j < i$. From the characterization of the equilibrium

$$\begin{aligned} U(j) - U(i) &= -\int_j^i \ell(x)u_1(x,x)dx \geq -\int_j^i \ell(i)u_1(x,x)dx \\ &\geq -\ell(i)\int_j^i u_1(x,i)dx = \ell(i)(u(j,i) - u(i,i)), \end{aligned}$$

where in the first equality we use the local ICs on the PAM CSE, in the first inequality we used $u_1(x, x) \le 0$ and $\ell(x) \ge \ell(i)$ for $x \le i$, in the second inequality we used supermodularity ($u_1(x, x) \le u_1(x, i)$ for $x \le i$) with $\ell(i) > 0$, and in the last equality we evaluated the integral. This shows us that the global downward ICs hold.

Existence of a PAM CSE. We still need to rule out the possibility that an (i, j) market is more profitable for $i \neq j$. First, assume that i and j are both strictly above $\underline{i}$, so in a proposed (i, j) market, we have a binding local incentive constraint on both sides. In the conjectured market, the local incentive constraints require

$$\lambda^a = \frac{U'(i)}{u_1(i, j)}, \ \lambda^b = \frac{U'(j)}{u_1(j, i)}.$$

Inverting the participation for type i and type j, we get:

$$\begin{aligned}\phi^a + \phi^b &= u(i, j) + u(j, i) - \frac{U(i)}{U'(i)} u_1(i, j) - \frac{U(j)}{U'(j)} u_1(j, i) \\ &= u(i, j) + u(j, i) - u(i, i)\frac{u_1(i, j)}{u_1(i, i)} - u(j, j)\frac{u_1(j, i)}{u_1(j, j)},\end{aligned}$$

where in the second equality, we use the equilibrium utility from PAM CSE. Under assumption 8(a), $\phi^a + \phi^b < 0$. Thus, there is no terms-of-trade attracting (i, j) and delivering non-negative payoffs to platforms.

Now consider a proposed $(\underline{i}, i)$ market with $i > \underline{i}$, where side a attracts type $\underline{i}$ and side b attracts type i. Type $\underline{i}$ is the lowest type and faces no downward incentive constraints, so the analysis differs from the previous case.

On side b, the downward incentive constraints require $U(k) - U(i) \ge \lambda^b(u(k, \underline{i}) - u(i, \underline{i}))$ for all $k < i$. With supermodularity and DWTP, $u_1(i, \underline{i}) < 0$ for all $i > \underline{i}$ (otherwise supermodularity would imply $u_1(i, j) > 0$ for $j > \underline{i}$, contradicting DWTP). The local IC on side b therefore pins down $\lambda^b = \tilde{\ell}(i) \equiv \ell(i)\, u_1(i, i)/u_1(i, \underline{i})$. The participation constraint gives the side-b fee $\tilde{\phi}(i) = u(i, \underline{i}) - u(i, i)\, u_1(i, \underline{i})/u_1(i, i)$. On side a, the participation constraint gives $\phi^a = u(\underline{i}, i) - \bar{\lambda}\, u(\underline{i}, \underline{i})/\lambda^a$. Their sum is

$$\phi^a + \phi^b = u(\underline{i}, i) + \tilde{\phi}(i) - \frac{\bar{\lambda}\, u(\underline{i}, \underline{i})}{\lambda^a}.$$

If $u(\underline{i}, i) + \tilde{\phi}(i) \le 0$, then $\phi^a + \phi^b \le 0$ for all $\lambda^a > 0$, so the market cannot generate positive fees. This is the first case in Assumption 8(b).

If $u(\underline{i}, i) + \tilde{\phi}(i) > 0$, positive fees require $\lambda^a > \bar{\lambda}\, u(\underline{i}, \underline{i})/(u(\underline{i}, i) + \tilde{\phi}(i))$. For the market to be both feasible and profitable, we need $(\lambda^a, \tilde{\ell}(i)) \in \mathbb{A}^o$ for some such λ^a. Assumption 8(b) rules this out by requiring $\big(\bar{\lambda}\, u(\underline{i}, \underline{i})/(u(\underline{i}, i) + \tilde{\phi}(i)),\, \tilde{\ell}(i)\big) \notin \mathbb{A}^o$. Since m is strictly decreasing in its first argument, any larger λ^a also has $(\lambda^a, \tilde{\ell}(i)) \notin \mathbb{A}^o$, making the market unprofitable or infeasible. This is the second case in Assumption 8(b).

The remaining CSE conditions follow from Proposition 1, Part 2: take $\nu^a = \nu^b$ to be the pushforward of $I\, dF$ on $\mathbb{I}$ under $i \mapsto \tau(i)$, set $d\mu = \ell(i)\, d\nu^s$, and verify Consistency (by construction, since $\Lambda^s(\tau(i)) = \ell(i)$), Market Clearing (with equality by construction), and Equilibrium Utility (since $U(i) = \ell(i)u(i,i)$ is achieved at $\tau(i)$). ■

D.2 Anchor-Matching Equilibrium

We characterize the equilibrium in which the lowest types have negative assortative matching and higher types have positive assortative matching. We also show how to verify global incentive constraints and the absence of profitable platform deviations.

Characterizing equilibrium. The equilibrium matching pattern parallels the IWTP case: there is a threshold type i^*; types $i \geq i^*$ match with the same type, and type $i < i^*$ matches with type $\sigma(i) > i^*$, the negatively-matched segment. For $i \geq i^*$, their equilibrium utility solves:

$$\frac{U'(i)}{U(i)} = \frac{u_1(i,i)}{u(i,i)}, \tag{24}$$

as in a PAM equilibrium with DWTP (Proposition 3), but with boundary condition $U(i^*) = \bar{\lambda}\, u(i^*, i^*)$. That is, in the positive assortative matching segment, the outcomes are the same as a PAM equilibrium where the lowest type is i^*.

To find i^* and $\sigma(i)$, we use the same pair of equations as in the IWTP case. In the negative assortative matching region, platforms earn zero profit with $c = 0$, which requires both zero platform matching probability and zero sum of fees, giving equations (20) and (21), respectively, which apply unchanged. For given i^*, $U(\sigma(i))$ and $U'(\sigma(i))$ are known since $\sigma(i) > i^*$ and so they must satisfy equation (24) for the positive assortative matching segment. From equation (21), we can recover $\sigma(i)$ given $U(i)$ and $U'(i)$. With the solution of $\sigma(i)$, we then solve equation (20) as an ODE for $U(i)$ with boundary condition $U(i^*) = \bar{\lambda}\, u(i^*, i^*)$. We again find i^* through the condition that $(\underline{i}, j)$ markets do not deliver positive profits as in the case of IWTP. This characterization is generic for any utility function with DWTP.

Unlike the IWTP example, where $u(i,j) = \sqrt{ij}$ yields a closed-form ODE for $\sigma(i)$ (equation 23), the functional form $u(i,j) = 1 - \kappa\, i(1-j)$ does not admit a tractable closed-form reduction: the positive assortative matching ODE (24) gives $U(\sigma(i))$ only as a non-algebraic function of $\sigma(i)$ and i^*, so the coupled system is solved numerically. In the upcoming steps, we verify the solved equilibrium satisfies all downward ICs and there are no profitable deviations, imposing $u(i,j) = 1 - \kappa\, i(1-j)$ and $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$.

Downward ICs. We first prove $U''(i) > 0$ for all i. In the positive assortative matching segment ($i \geq i^*$), $U'(i) = \ell(i)\, u_1(i,i) = -\kappa(1-i)\ell(i)$, and differentiating gives

$$U''(i) = \kappa\ell(i) - \kappa(1-i)\ell'(i) = \frac{\kappa\ell(i)}{u(i,i)} > 0,$$

where we use $\ell'(i)/\ell(i) = -u_2(i,i)/u(i,i) = -\kappa i/u(i,i)$.

In the negatively-matched segment ($i < i^*$), $U'(i) = -\kappa(1-\sigma(i))\ell^a(i)$, so

$$U''(i) = \kappa(1-\sigma(i)) \left[\frac{\ell^a(i)\sigma'(i)}{1-\sigma(i)} - \ell^{a\prime}(i) \right].$$

Imposing the functional form on the zero-fee condition (21) and using $U(\sigma(i))/U'(\sigma(i)) = u(\sigma(i),\sigma(i))/u_1(\sigma(i),\sigma(i))$ from the positive assortative matching segment yields

$$\frac{U(i)}{\ell^a(i)} = 2 - \kappa\, i(1-\sigma(i)) - \frac{1-i}{1-\sigma(i)}. \tag{25}$$

Differentiating both sides and using $U'(i)/\ell^a(i) = u_1(i,\sigma(i)) = -\kappa(1-\sigma(i))$ to simplify the left-hand side gives

$$\frac{\ell^a(i)\sigma'(i)}{1-\sigma(i)} - \ell^{a\prime}(i) = \frac{\ell^a(i)^2}{U(i)} \left[\frac{2\sigma'(i)(i-\sigma(i))}{(1-\sigma(i))^2} + \frac{1}{1-\sigma(i)} \right].$$

Since $\sigma'(i) < 0$ and $i < \sigma(i)$, we have $\sigma'(i)(i-\sigma(i)) > 0$, so the bracketed expression is positive. This gives $U''(i) > 0$.

Next, consider the incentive constraint keeping k out of the s side of the (i,j) market:

$$U(k) - U(i) \geq \lambda^s(u(k,j) - u(i,j)) = -\lambda^s\kappa(k-i)(1-j)$$

Using the local incentive constraint $\lambda^s = U'(i)/u_1(i,j) = -U'(i)/\kappa(1-j)$, this reduces to

$$U(k) - U(i) \geq U'(i)(k-i).$$

Strict convexity of U implies this is a strict inequality.

No profitable deviations. We now show there are no profitable deviations, dividing the possible deviations (i,j) into four cases: (1) $i,j \geq i^*$; (2) $\underline{i} < i < i^*$ and $j \geq \sigma(i)$; (3)

$\underline{i} < i < i^*$ and $i^* \leq j < \sigma(i)$; and (4) $\underline{i} < i, j < i^*$. We demonstrate the proofs for the third and fourth cases under the specification $\kappa = 2$ and $\gamma = 1$.

In the first case, both types are in the positive assortative matching segment, so $U(i)/U'(i) = u(i,i)/u_1(i,i)$ and likewise for j. The sum of fees in a proposed (i,j) market is therefore

$$\Psi(i,j) = u(i,j) + u(j,i) - \frac{u(i,i)}{u_1(i,i)}u_1(i,j) - \frac{u(j,j)}{u_1(j,j)}u_1(j,i) = -\frac{(i-j)^2}{(1-i)(1-j)} < 0.$$

In the second case, the sum of fees in a proposed (i,j) market is

$$\Psi(i,j) = u(i,j) + u(j,i) - \frac{U(i)}{U'(i)}u_1(i,j) - \frac{u(j,j)}{u_1(j,j)}u_1(j,i),$$

where we use $U(j)/U'(j) = u(j,j)/u_1(j,j)$ since $j \geq i^*$ is in the positive assortative matching segment. Note that $U(i)/U'(i)$ depends on i's equilibrium allocation (and hence on $\sigma(i)$) but not on the deviation partner j. Imposing $u(i,j) = 1 - \kappa\, i(1-j)$ and differentiating with respect to j,

$$\Psi_2(i,j) = \kappa\, i - \frac{U(i)}{U'(i)}\kappa - \frac{1-i}{(1-j)^2}.$$

From equation (25) and the local IC $U'(i) = \ell^a(i)u_1(i,\sigma(i))$, we get $\kappa\, U(i)/U'(i) = \kappa\, i - 2/(1-\sigma(i)) + (1-i)/(1-\sigma(i))^2$, so

$$\Psi_2(i,j) = \frac{2}{1-\sigma(i)} - \frac{1-i}{(1-\sigma(i))^2} - \frac{1-i}{(1-j)^2}.$$

Since $\Psi(i,\sigma(i)) = 0$, $\Psi_2(i,\sigma(i)) = -\frac{2(\sigma(i)-i)}{(1-\sigma(i))^2} < 0$, and $\Psi_{2,2}(i,j) < 0$, this proves $\Psi(i,j) < 0$ for all $j > \sigma(i)$.

For the final two cases, as well as the verification when $i = \underline{i}$, we proceed numerically for our example.

Market clearing. As in the IWTP case, for type $i < i^*$, feasibility of the negative assortative matching segment requires

$$\frac{\ell^a(i)}{\ell^b(i)} \leq -\sigma'(i),$$

with the remaining agents of type $\sigma(i)$ absorbed by the positive assortative matching terms-of-trade. Unlike the IWTP case, where $\ell^a/\ell^b < 1$, here $\ell^a/\ell^b > 1$ since the matching probability $\ell^a(i)$ is decreasing in the DWTP environment. We verify this condition numerically along with the equilibrium computation.

E Proofs and Supplements to Section 7

This appendix proves the two main results in Section 7. Section E.1 characterizes a PAM CSE with asymmetry and zero costs under IWTP. Section E.2 proves Proposition 4, characterizing PAM with symmetry and positive costs. We denote the support of the type distribution on side s by $\mathbb{I}^s = [\underline{i}^s, \bar{i}^s]$, with density f^s.

E.1 Asymmetry with Zero Costs

Characterization 1 (PAM with Asymmetry and Zero Costs under IWTP) *Assume Common Ranking, Supermodularity, Limit Supermodularity, and IWTP. Also assume $c = 0$. In a PAM equilibrium, for the terms-of-trade attracting types $(i, \sigma(i))$, denote the side-a and side-b matching probabilities as $\ell^a(i)$ and $\ell^b(i)$. The following equation system characterizes the equilibrium outcomes:*

$$m(\ell^a(i), \ell^b(i)) = 0,$$

$$U^{a\prime}(i) = \ell^a(i)\, u_1^a(i, \sigma(i)), \qquad U^{b\prime}(\sigma(i)) = \ell^b(i)\, u_1^b(\sigma(i), i),$$

$$U^a(i) = \ell^a(i)\, (u^a(i, \sigma(i)) - \Phi^a(i)), \qquad U^b(\sigma(i)) = \ell^b(i)\, (u^b(\sigma(i), i) - \Phi^b(i)),$$

$$\frac{\ell^{a\prime}(i)}{\ell^a(i)} + \frac{u_{12}^a(i, \sigma(i))}{u_1^a(i, \sigma(i))}\sigma'(i) = \frac{m_2\big(\ell^a(i), \ell^b(i)\big)\, \ell^b(i)}{m_1\big(\ell^a(i), \ell^b(i)\big)\, \ell^a(i)} \frac{u_{12}^b(\sigma(i), i)}{u_1^b(\sigma(i), i)},$$

$$\sigma'(i) = \frac{I^a f^a(i)\, \ell^a(i)}{I^b f^b(\sigma(i))\, \ell^b(i)},$$

with boundary conditions $\sigma(\underline{i}^a) = \underline{i}^b$, $\sigma(\bar{i}^a) = \bar{i}^b$, and $\Phi^a(\underline{i}^a) + \Phi^b(\underline{i}^b) = 0$.

Proof of Characterization 1. We derive each equation in the characterization in turn.

Zero platform matching probability. From Lemma 4, $m(\ell^a(i), \ell^b(i)) = 0$ at μ-a.e. active terms-of-trade. This is the first equation in the characterization.

Local ICs, participation constraint, and platform value. For a platform attracting the pair (i, j), the local ICs (Lemma 3) pin down matching probabilities $\lambda^a(i, j) = U^{a\prime}(i)/u_1^a(i, j)$ and $\lambda^b(i, j) = U^{b\prime}(j)/u_1^b(j, i)$. At the equilibrium pair we write $\ell^a(i) \equiv \lambda^a(i, \sigma(i))$ and $\ell^b(i) \equiv \lambda^b(i, \sigma(i))$, the second line in the characterization. The third line gives the participation constraints.

The platform value $\hat{V}(i, j)$ is given by equation (10):

$$\hat{V}(i, j) = m(\lambda^a(i, j),\, \lambda^b(i, j)) \cdot S(i, j),$$

where $S(i,j) \equiv u^a(i,j) + u^b(j,i) - U^a(i)/\lambda^a(i,j) - U^b(j)/\lambda^b(i,j)$ is the fee sum.

Platform optimality. In equilibrium, $\hat{V}(i,\sigma(i)) = 0 = c$ for all i, and $\hat{V}(i,j) \leq 0$ for all (i,j). Since $\hat{V}$ is maximized along the equilibrium path, the partial derivative with respect to its first argument, holding j fixed, vanishes: $\hat{V}_1(i,\sigma(i)) = 0$.

Applying the product rule to $\hat{V} = m \cdot S$:

$$\hat{V}_1 = \left(m_1 \frac{\partial \lambda^a}{\partial i} + m_2 \frac{\partial \lambda^b}{\partial i} \right) \cdot S \quad + \quad m \cdot S_1.$$

Evaluating at $(i,\sigma(i))$, the second term vanishes because $m(\ell^a(i), \ell^b(i)) = 0$. Assuming $S(i,\sigma(i)) \neq 0$, we can divide through to obtain

$$0 = m_1(\ell^a(i), \ell^b(i)) \left. \frac{\partial \lambda^a}{\partial i} \right|_{j=\sigma(i)} + m_2(\ell^a(i), \ell^b(i)) \left. \frac{\partial \lambda^b}{\partial i} \right|_{j=\sigma(i)}. \tag{26}$$

We now compute the two partial derivatives. Since $\lambda^b(i,j) = U^{b\prime}(j)/u^b_1(j,i)$ and only the denominator depends on i:

$$\left. \frac{\partial \lambda^b}{\partial i} \right|_{j=\sigma(i)} = -\ell^b(i) \, \frac{u^b_{12}(\sigma(i), i)}{u^b_1(\sigma(i), i)}.$$

For $\partial \lambda^a / \partial i$, note that $\lambda^a(i,j) = U^{a\prime}(i)/u^a_1(i,j)$ depends on i through both the numerator and the denominator, making $U^{a\prime\prime}(i)$ appear. To avoid working with $U^{a\prime\prime}$ directly, we use the identity $\ell^a(i) = \lambda^a(i,\sigma(i))$ and totally differentiate:

$$\ell^{a\prime}(i) = \left. \frac{\partial \lambda^a}{\partial i} \right|_{j=\sigma(i)} + \left. \frac{\partial \lambda^a}{\partial j} \right|_{j=\sigma(i)} \sigma'(i).$$

Since $\frac{\partial \lambda^a}{\partial j}\Big|_{j=\sigma(i)} = -\ell^a(i) \, \frac{u^a_{12}(i,\sigma(i))}{u^a_1(i,\sigma(i))}$, solving gives

$$\left. \frac{\partial \lambda^a}{\partial i} \right|_{j=\sigma(i)} = \ell^{a\prime}(i) + \ell^a(i) \, \frac{u^a_{12}(i,\sigma(i))}{u^a_1(i,\sigma(i))} \, \sigma'(i).$$

Substituting both derivatives into (26) and dividing by $m_1(\ell^a(i), \ell^b(i)) \, \ell^a(i)$ yields the fourth line in the characterization.

Market clearing. In PAM, side-a type i matches with side-b type $\sigma(i)$, where σ is increasing. Consistency of the equilibrium measures on the active curve, $d\mu = \ell^s \, d\nu^s$, implies $d\nu^b/d\nu^a = \ell^a/\ell^b$: at each terms-of-trade, the ratio of side-b to side-a agent flow equals the inverse ratio of their matching probabilities. Equating accumulated side-b agents used by side-a types in $[\underline{i}^a, i]$ with available supply in $[\underline{i}^b, \sigma(i)]$:

$$I^b F^b(\sigma(i)) = \int_{\underline{i}^a}^{i} \frac{\ell^a(x)}{\ell^b(x)}\, I^a \, dF^a(x).$$

Differentiating both sides:

$$\sigma'(i) = \frac{I^a f^a(i)\, \ell^a(i)}{I^b f^b(\sigma(i))\, \ell^b(i)} > 0.$$

This delivers the final line in the characterization and confirms that σ is increasing, as required for PAM. The derivation remains valid as long as $\ell^b > 0$; only the ratio ℓ^a/ℓ^b enters, so the argument is robust to $\ell^a \to 0$ in limiting cases.

Boundary conditions. Market clearing at the endpoints requires $\sigma(\underline{i}^a) = \underline{i}^b$ and $\sigma(\bar{i}^a) = \bar{i}^b$. The $(\underline{i}^a, \underline{i}^b)$ market is unconstrained (no downward ICs for the lowest types). Since $c = 0$ and $m = 0$, the platform breaks even regardless of fees, so the zero-fee condition $\Phi^a(\underline{i}^a) + \Phi^b(\underline{i}^b) = 0$ pins down the level. ∎

E.2 Positive Costs with Symmetry

Proof of Proposition 4.

Setup. Any symmetric PAM CSE must solve problem (9) with value c at every (i, i) market. Since $c > 0$, we can ignore the nonnegativity constraint on fees and then eliminate fees via the participation constraint:

$$c = \max_{\lambda \in [0, \bar{\lambda}]} 2m(\lambda, \lambda) \left(u(i,i) - \frac{U(i)}{\lambda} \right), \qquad \text{s.t. } U'(i) = \lambda\, u_1(i,i). \tag{27}$$

Unconstrained markets. Start with an unconstrained version of the problem. Using $n = m(\lambda, \lambda)/\lambda$, we can rewrite the unconstrained problem as

$$c = \max_{n \geq 0} 2\big(M^*(n)\, u(i,i) - n\, U(i)\big), \tag{28}$$

where $M^*(n) \equiv M(n,n)$. Since M is strictly concave, this has a unique maximum satisfying $M^{*\prime}(n)\, u(i,i) = U(i)$, and the free-entry condition becomes $c/(2u(i,i)) = M^*(n) - nM^{*\prime}(n)$. The right-hand side is increasing (by concavity), zero at $n = 0$, and approaches $m(0,0)$ as $n \to \infty$. Since $2u(i,i)m(0,0) > c$ by assumption (13), there is a unique solution $n^*(i) > 0$, from which we recover $U^*(i)$, $\ell^*(i) = L(n^*(i), n^*(i))$, and $\Phi^*(i) > 0$.

For the lowest type $i = \underline{i}$, the incentive constraint is slack, so this pins down $U(\underline{i}) = U^*(\underline{i})$ and $\ell(\underline{i}) = \ell^*(\underline{i})$.

Constrained markets. For types where incentive constraint binds, the objective in (27) is decreasing in $U(i)$, so $U(i) > U^*(i)$ is inconsistent with an active market. Thus $U(i) \leq U^*(i)$. If the IC binds strictly at i, the allocation is different than the unconstrained optimum, so $U(i) < U^*(i)$. Concavity of M^* then implies there are exactly two values of λ satisfying the free-entry condition: one in $(\ell^*(i), \bar{\lambda}]$ and one in $(0, \ell^*(i))$.

This also implies that $h(\lambda, i)$ is single-peaked in λ, with maximum $U^*(i)$ at $\ell^*(i)$. To see this, note that $h(\lambda, i) = U$ if and only if λ solves the free entry condition

$$c = 2m(\lambda, \lambda)\left(u(i, i) - \frac{U}{\lambda}\right)$$

at utility level U. Since for each $U < U^*$, there are exactly two such solutions, one above $\ell^*(i)$ and one below, the level sets of h are two-point sets below the maximum and a singleton at the maximum, which is single-peakedness.

At each solution, the free-entry condition reads $U(i) = h(\ell(i), i)$, where h is defined in equation (12). Differentiating:

$$U'(i) = h_1(\ell(i), i)\, \ell'(i) + \ell(i)(u_1(i, i) + u_2(i, i)).$$

Imposing the local IC $U'(i) = \ell(i)\, u_1(i, i)$ gives the ODE (14).

Increasing Willingness to Pay. Now assume IWTP. For $j < i$, $u(i, i) - u(j, i) > 0$ and so the downward IC gives a lower bound on λ, $\lambda \geq \sup_{j<i}(U(i) - U(j))/(u(i, i) - u(j, i))$. If $\ell(i)$ were the smaller solution (below $\ell^*(i)$), then $\ell^*(i)$ would be feasible and yield value exceeding c, a contradiction. So $\ell(i) \in (\ell^*(i), \bar{\lambda}]$.

Continuity of $U(i)$ (Lemma 8) and the choice of root $\ell(i) > \ell^*(i)$ ensure $\ell(i)$ is continuous. For any active τ attracting type k^s on side s, the expected payoff $\bar{U}^s(i, \tau, \lambda^s) = \lambda^s(u^s(i, k^{\bar{s}}) - \phi^s)$ is Lipschitz in i with constant $L \equiv \sup_{\mathbb{I}^a \times \mathbb{I}^b} |u_1^s|$, finite by continuous differentiability of u^s in its first argument on the compact set $\mathbb{I}^a \times \mathbb{I}^b$ and since $\lambda^s \leq 1$. The upper envelope $U^s(\cdot) = \max\{0, \sup_\tau \bar{U}^s(\cdot, \tau, \Lambda^s(\tau))\}$ inherits the same Lipschitz constant, hence is absolutely continuous. Combined with continuity of ℓ and the a.e. identity $U'(i) = \ell(i)u_1(i, i)$ from Lemma 3, absolute continuity upgrades this identity to pointwise equality, so the ODE (14) holds everywhere.

The local IC gives $U'(i) = \ell(i)\, u_1(i, i) > 0$, so U is increasing. Since $U(\underline{i}) > 0$, we have $U(i) > 0$ for all i. Since $\ell(i) > \ell^*(i)$ and h is single-peaked at $\ell^*(i)$, we have $h_1(\ell(i), i) < 0$.

Thus equation (14) gives $\ell'(i) > 0$. Since ℓ is strictly increasing, $m(\ell(i), \ell(i))$ is strictly decreasing, and the free-entry condition $2m(\ell(i), \ell(i))\,\Phi(i) = c$ implies $\Phi(i)$ is strictly increasing.

Finally, we turn to the downward incentive constraints, $U(j)-U(i) \geq \ell(i)(u(j,i)-u(i,i))$ for $j < i$. Using the local IC:

$$U(j) - U(i) = -\int_j^i \ell(x)\, u_1(x,x)\, dx.$$

Since ℓ is increasing, $\ell(x) \leq \ell(i)$ for $x \in [j,i]$. By supermodularity, $u_1(x,x) \leq u_1(x,i)$ for $x < i$. Both terms are positive under IWTP, so $\ell(x)\, u_1(x,x) \leq \ell(i)\, u_1(x,i)$ and therefore

$$-\int_j^i \ell(x)\, u_1(x,x)\, dx \geq -\ell(i) \int_j^i u_1(x,i)\, dx = \ell(i)(u(j,i) - u(i,i)),$$

verifying the global downward IC.

Decreasing Willingness to Pay. Now assume DWTP. The incentive constraints now imply an upper bound $\lambda \leq \sup_{j<i}(U(j) - U(i))/(u(j,i) - u(i,i))$, which rules out the upper solution. So $\ell(i) \in (0, \ell^*(i))$. Continuity of U again implies continuity of ℓ, and Lipschitz continuity of U (as in the IWTP case) implies $U'(i) = \ell(i)u_1(i,i)$ holds pointwise, so the ODE (14) holds everywhere.

Now $U'(i) = \ell(i)\, u_1(i,i) < 0$ (since $u_1(i,i) < 0$ under DWTP), so U is decreasing. Since $\ell(i) < \ell^*(i)$, single-peakedness gives $h_1(\ell(i), i) > 0$, and equation (14) gives $\ell'(i) < 0$. Finally, ℓ decreasing implies Φ decreasing through the free-entry condition $c = 2m(\ell(i), \ell(i))\,\Phi(i)$.

Turning to the downward incentive constraints, $U(j) - U(i) \geq \ell(i)(u(j,i) - u(i,i))$ for $j < i$, we note that both sides are positive. Using the local IC we have

$$U(j) - U(i) = -\int_j^i \ell(x)\, u_1(x,x)\, dx.$$

Then since $\ell(x) \geq \ell(i) > 0$ and $-u_1(x,x) \geq -u_1(x,i) > 0$ (supermodularity) for $x \in [j,i]$, we get

$$-\int_j^i \ell(x)\, u_1(x,x)\, dx \geq -\ell(i) \int_j^i u_1(x,i)\, dx = \ell(i)(u(j,i) - u(i,i)),$$

verifying the global IC. ■

Online Appendix for "Competitive Sorting with Bilateral Private Information"

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F Existence of Competitive Search Equilibrium

F.1 Statement and Roadmap

We first define strict IWTP:

Assumption 9 (Strict IWTP) $\forall\, s \in \{a, b\}\; i > i' \in \mathbb{I}^s,\; j \in \mathbb{I}^{\bar{s}},\; u^s(i, j) > u^s(i', j)$.

Combining this with Common Ranking, Supermodularity, a finite type space, and a positive intermediation cost delivers a general existence theorem:[21]

Theorem 1 *Assume Common Ranking, Supermodularity, and strict IWTP. If $c > 0$ and $\mathbb{I}^a, \mathbb{I}^b$ are finite, a CSE exists.*

The proof uses a Kakutani fixed-point argument on the pair (U, μ), where μ is the match measure of Definition 2. At given equilibrium utility U, platforms choose μ to concentrate match mass on profit-maximizing markets, and a Walrasian auctioneer adjusts U in the direction of excess demand, in the spirit of Gale (1955), Debreu (1956), and Nikaido (1956). The technical crux is that the platform's value $\hat{V}(k^a, k^b; U)$ from attracting (k^a, k^b) is continuous in U up to the boundary where the market becomes infeasible, where $\hat{V} \to 0$.

Strict IWTP is what delivers boundary continuity. Because $u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}}) > 0$ for every $i < k^s$, downward incentive constraints act uniformly as a strictly positive lower bound on λ^s. Any way the feasible set becomes empty therefore drives either $m \to 0$ or $\phi^a + \phi^b \to 0$, with $\hat{V} \to 0$ in either case. The Kakutani argument then runs cleanly.

Outside IWTP, this boundary continuity fails, and existence can fail with it. The failures take qualitatively different forms. Under strict DWTP, the coefficient $u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}})$ is uniformly negative, so downward incentive constraints act as an upper bound on λ^s.

[21] The finite-type assumption delivers compactness. Extending the proof to a continuous type space would require additional functional-analytic machinery beyond the scope of this paper. Sections 5–7 of the main text construct continuous-type equilibria directly and do not rely on Theorem 1.

At $U^s(k^s) = 0$, this upper bound collapses to zero, but the platform retains a profitable degenerate deviation: a market with $\lambda^s = 0$, fees set to the rationality bound, and matching probability $m(0,0)$, earning value $m(0,0)f(k^a, k^b) > 0$. Any perturbation of $U^s(k^s)$ off zero destroys this deviation, and the resulting discontinuity in $\hat{V}$ at the corner of $\mathcal{U}$ defeats the fixed-point argument. Section F.5 exhibits a two-type symmetric example in which no CSE exists for an open interval of intermediation costs.[22] The corner-driven character of this failure suggests that additional structure ruling out boundary equilibria, such as a participation condition guaranteeing positive utility for all types, may rescue existence under DWTP. We leave a general result in this direction to future work.

Under non-monotone WTP, the failure is interior rather than boundary. With $u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}})$ taking different signs across types i, the downward incentive constraints from different types impose conflicting bounds on λ^s, lower bounds from some types and upper bounds from others, that can cross in the interior of $\mathbb{\Lambda}$. The discontinuity in $\hat{V}$ then occurs at strictly positive utility values rather than at the corner of $\mathcal{U}$, and the mechanism is robust to perturbations of the example. Section F.6 exhibits a three-type symmetric example. We see no analogous structural fix.

Strictness of IWTP, positivity of c, and finiteness of the type space play distinct roles in the proof, which we summarize before turning to the construction. The proof establishes existence in four steps: (i) reduce the analysis of active markets to a finite-dimensional program indexed by the target types (k^a, k^b), with value $\hat{V}(k^a, k^b; U)$; (ii) define a platform correspondence (which chooses μ given U) and a Walrasian-auctioneer correspondence (which adjusts U in the direction of excess demand); (iii) apply Kakutani's theorem to obtain a fixed point (U^*, μ^*); (iv) verify that (U^*, μ^*) satisfies Definition 2. The three assumptions of the theorem play distinct roles. Strict IWTP makes the downward incentive constraints a *uniform* positive lower bound on λ^s, which is what delivers continuity of $\hat{V}$ up to the boundary of $\mathcal{U}$ (Lemma 11). A positive c ensures that $A(U)$ (the set of free-entry markets at U) is closed and that $m > 0$ at every supported market when $V^*(U^*) = c$. Finiteness of $\mathbb{K}^a$, $\mathbb{K}^b$ makes the domain $\mathcal{U}$ a finite-dimensional polytope and the measure space $\mathcal{M}$ compact.

F.2 Setup

Fix a constant $C > \max_{i \in \mathbb{K}^a,\, j \in \mathbb{K}^b} \left[u^a(i,j) + u^b(j,i) \right]$ and set

$$\mathcal{U} \equiv [0, C]^{|\mathbb{K}^a|} \times [0, C]^{|\mathbb{K}^b|},$$

[22] The applications in Sections 6 and 7.2 of the main text construct DWTP equilibria directly and do not rely on Theorem 1.

a compact convex polytope. We will prove existence of a CSE with equilibrium utility $U \in \mathcal{U}$.

Next, fix $(k^a, k^b) \in \mathbb{I}^a \times \mathbb{I}^b$ and $U \in \mathcal{U}$. By Proposition 1, a separating terms-of-trade attracting (k^a, k^b) solves

$$\hat{V}(k^a, k^b; U) \;=\; \sup_{\lambda^a, \lambda^b, \phi^a, \phi^b} \; m(\lambda^a, \lambda^b)(\phi^a + \phi^b) \qquad (29)$$
$$\text{subject to } (\lambda^a, \lambda^b, \phi^a, \phi^b) \in L(k^a, k^b; U),$$

where $L(k^a, k^b; U) \subset \mathbb{R}^4$ denotes the feasible set, the combinations of $(\lambda^a, \lambda^b, \phi^a, \phi^b)$ satisfying

$$\lambda^s\big(u^s(k^s, k^{\bar{s}}) - \phi^s\big) = U^s(k^s), \quad s = a, b; \qquad \text{(P)}$$
$$U^s(i) \geq \lambda^s\big(u^s(i, k^{\bar{s}}) - \phi^s\big), \quad i < k^s,\; s = a, b; \qquad \text{(IC)}$$
$$\phi^a + \phi^b \geq 0; \qquad \text{(F)}$$
$$\phi^s \leq u^s(k^s, k^{\bar{s}}), \quad s = a, b; \qquad \text{(B)}$$
$$(\lambda^a, \lambda^b) \in \mathbb{\Lambda}.$$

The bounds (F) and (B) confine ϕ^s to the compact interval $[-u^{\bar{s}}(k^{\bar{s}}, k^s),\, u^s(k^s, k^{\bar{s}})]$ and $\mathbb{\Lambda}$ is compact, so L sits inside a compact set. When $L(k^a, k^b; U) = \emptyset$, no feasible terms-of-trade attracts (k^a, k^b), and we adopt the convention $\hat{V}(k^a, k^b; U) = 0$: this is consistent with the platform's outside option of not operating this market (which earns gross profit zero), and makes $\hat{V}(\cdot; U)$ upper semicontinuous up to the boundary of the feasibility region (Lemma 11).

For $\lambda^s > 0$, use (P) to eliminate ϕ^s from the remaining constraints:

$$U^s(k^s) - U^s(i) \;\leq\; \lambda^s\big[u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}})\big], \quad i < k^s,\; s = a, b; \qquad \text{(IC}'\text{)}$$
$$\frac{U^a(k^a)}{\lambda^a} + \frac{U^b(k^b)}{\lambda^b} \;\leq\; f(k^a, k^b) \;\equiv\; u^a(k^a, k^b) + u^b(k^b, k^a); \qquad \text{(F}'\text{)}$$

and (B) reduces to $U^s(k^s)/\lambda^s \geq 0$, automatic. At $\lambda^s = 0$, (P) forces $U^s(k^s) = 0$ and ϕ^s is free within $[-u^{\bar{s}}(k^{\bar{s}}, k^s),\, u^s(k^s, k^{\bar{s}})]$.

Under strict IWTP, the coefficient $u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}})$ on the right side of (IC$'$) is strictly positive for every $i < k^s$, so (IC$'$) acts uniformly on each side as a strictly positive lower bound on λ^s. This uniform-sign property, which would fail under weak IWTP or under non-Monotone WTP, is what drives the continuity arguments below.

F.3 Continuity of $\hat{V}$

This subsection establishes the technical core of the existence proof: the platform's value function V^* is continuous on $\mathcal{U}$, and the argmax in (29) has closed graph. Throughout, we maintain Common Ranking, Supermodularity, and strict IWTP.

Lemma 11 *Assume Common Ranking, Supermodularity, and strict IWTP. Then* $V^*(U) \equiv \max_{(k^a,k^b)\in\mathbb{I}^a\times\mathbb{I}^b} \hat{V}(k^a,k^b;U)$ *is continuous on* $\mathcal{U}$*, each* $\hat{V}(k^a,k^b;\cdot)$ *is upper semicontinuous on* $\mathcal{U}$*, and for every* $U\in\mathcal{U}$ *with* $L(k^a,k^b;U)\neq\emptyset$*, the set of maximizers in Problem (29) is nonempty and has closed graph in* U*.*

The proof proceeds through two intermediate lemmas: Lemma 12 establishes that the feasibility correspondence has closed graph, and Lemma 13 establishes that locally feasible points at one U can be perturbed to stay feasible at nearby U'. Together they deliver continuity of V^*.

Lemma 12 *The graph* $\{(U,\lambda,\phi)\in\mathcal{U}\times\mathbb{R}^4 : (\lambda,\phi)\in L(k^a,k^b;U)\}$ *is closed. For every* $U\in\mathcal{U}$ *with* $L(k^a,k^b;U)\neq\emptyset$*, the set of solutions to problem (29) is nonempty, compact, and has closed graph in* U*. Moreover,* $\hat{V}(k^a,k^b;\cdot)$ *is upper semicontinuous on* $\mathcal{U}$*.*

Proof of Lemma 12. The constraint $(\lambda^a,\lambda^b)\in\mathbb{\Lambda}$ cuts out a closed subset of $\mathcal{U}\times\mathbb{R}^4$ because $\mathbb{\Lambda}$ is closed. Each of (P), (IC), (F), (B) is a continuous (in)equality in (λ,ϕ,U), hence cuts out a closed subset. The graph is the intersection of these closed sets, hence closed.

The compact envelope $\mathbb{\Lambda}\times\prod_s[-u^{\bar{s}}(k^{\bar{s}},k^s),u^s(k^s,k^{\bar{s}})]$ makes $L(k^a,k^b;U)$ compact for each U, and the objective $m(\lambda)(\phi^a+\phi^b)$ is continuous, so the argmax is nonempty (Weierstrass) and closed. Closed-graph property of the argmax follows from closed graph of L plus continuity of the objective by Berge's theorem.

For upper semicontinuity of $\hat{V}(k^a,k^b;\cdot)$: let $U_n\to U$. If infinitely many U_n have $L(k^a,k^b;U_n)=\emptyset$, then $\hat{V}(k^a,k^b;U_n)=0\leq\hat{V}(k^a,k^b;U)$ along that subsequence. For the remaining U_n, pick maximizers $(\lambda_n,\phi_n)\in L(U_n)$; a convergent subsequence has limit $(\lambda_*,\phi_*)\in L(U)$ by closed graph, hence $\hat{V}(k^a,k^b;U_n)\to m(\lambda_*)(\phi^a_*+\phi^b_*)\leq\hat{V}(k^a,k^b;U)$. So $\limsup\hat{V}(k^a,k^b;U_n)\leq\hat{V}(k^a,k^b;U)$. ■

Lemma 13 *Let* $U\in\mathcal{U}$*,* $(k^a,k^b)\in\mathbb{I}^a\times\mathbb{I}^b$*, and* $(\lambda^*,\phi^*)\in L(k^a,k^b;U)$ *with* $m(\lambda^*)>0$ *and* $\phi^{*a}+\phi^{*b}>0$*. For* $\epsilon>0$*, set* $\lambda'^s=\max\{\lambda^{*s},\epsilon\}$ *on each side, and* $\phi'(U'')^s=u^s(k^s,k^{\bar{s}})-U''^s(k^s)/\lambda'^s$ *for* $U''\in\mathcal{U}$*. Under strict IWTP, for* $\epsilon>0$ *sufficiently small:*

1. $(\lambda',\phi'(U))\in L(k^a,k^b;U)$*;*
2. $m(\lambda')(\phi'(U)^a+\phi'(U)^b)\to m(\lambda^*)(\phi^{*a}+\phi^{*b})$ *as* $\epsilon\to 0$*;*
3. *there exists a neighborhood* N *of* U *in* $\mathcal{U}$ *such that* $(\lambda',\phi'(U''))\in L(k^a,k^b;U'')$ *for every* $U''\in N$*.*

Proof of Lemma 13. $\lambda^* \in \mathbb{A}$ with $m(\lambda^*) > 0$ and m continuous strictly decreasing imply λ^* lies in the set $\{m > 0\} \cap \mathbb{A}$. For ϵ small, λ' differs from λ^* only on sides with $\lambda^{*s} = 0$, and only by a small amount ϵ, so $\lambda' \in \mathbb{A}$ with $m(\lambda') > 0$. We verify each constraint at $(\lambda', \phi'(U), U)$.

(P) *at* (k^a, k^b). Holds by construction of $\phi'(U)$: $\lambda'^s(u^s(k^s, k^{\bar{s}}) - \phi'(U)^s) = \lambda'^s \cdot U^s(k^s)/\lambda'^s = U^s(k^s)$.

(F) *and* (B). On sides with $\lambda^{*s} > 0$: $\lambda'^s = \lambda^{*s}$ and $\phi'(U)^s = \phi^{*s}$ (identical to the original maximizer), so these constraints hold as at (λ^*, ϕ^*). On sides with $\lambda^{*s} = 0$: (P) at (λ^*, ϕ^*) forces $U^s(k^s) = 0$, so $\phi'(U)^s = u^s(k^s, k^{\bar{s}})$, saturating (B) with equality. (F) then reads $u^s(k^s, k^{\bar{s}}) + \phi'(U)^{\bar{s}} \geq 0$; on side $\bar{s}$, $\phi'(U)^{\bar{s}}$ equals $\phi^{*\bar{s}}$ if $\lambda^{*\bar{s}} > 0$, or $u^{\bar{s}}(k^{\bar{s}}, k^s)$ if $\lambda^{*\bar{s}} = 0$. In the first subcase, $\phi^{*a} + \phi^{*b} > 0$ gives $\phi^{*\bar{s}} > -u^s(k^s, k^{\bar{s}})$ (from (F) at (λ^*, ϕ^*) when ϕ^{*s} equals its upper bound $u^s(k^s, k^{\bar{s}})$), so (F) holds. In the second subcase, $\phi'(U)^s + \phi'(U)^{\bar{s}} = u^s(k^s, k^{\bar{s}}) + u^{\bar{s}}(k^{\bar{s}}, k^s) = f(k^s, k^{\bar{s}}) > 0$.

(IC) *at* (k^a, k^b)*, side* s*, type* $i < k^s$. Under strict IWTP, $c_i \equiv u^s(k^s, k^{\bar{s}}) - u^s(i, k^{\bar{s}}) > 0$. Via (IC′), the IC is $U^s(k^s) - U^s(i) \leq \lambda'^s \cdot c_i$. When $\lambda^{*s} > 0$: $\lambda'^s = \lambda^{*s}$, and the IC at (λ^*, ϕ^*) gives the result directly. When $\lambda^{*s} = 0$: $U^s(k^s) = 0$, so the IC reads $-U^s(i) \leq \epsilon c_i$, i.e., $U^s(i) \geq -\epsilon c_i$. This holds because $U^s(i) \geq 0$ (box constraint on $\mathcal{U}$).

Extension to neighborhood. The above establishes (1); for (3), fix ϵ small and consider $U'' \in \mathcal{U}$ close to U. (P) at $(k^a, k^b, U'', \lambda', \phi'(U''))$ holds by construction. For (F) and (B): on sides with $\lambda^{*s} > 0$, $\phi'(U'')^s \to \phi^{*s}$ continuously as $U'' \to U$, so the constraints hold for U'' close enough. On sides with $\lambda^{*s} = 0$: $\phi'(U'')^s = u^s(k^s, k^{\bar{s}}) - U''^s(k^s)/\epsilon$. As $U'' \to U$ (with $U^s(k^s) = 0$), $U''^s(k^s) \to 0$, so $\phi'(U'')^s \to u^s(k^s, k^{\bar{s}})$; (B) holds with small slack, and (F) reads $\phi'(U'')^s + \phi'(U'')^{\bar{s}} \to f(k^s, k^{\bar{s}}) > 0$. For (IC): the IC at U'' reads $U''^s(k^s) - U''^s(i) \leq \lambda'^s \cdot c_i$. When $\lambda^{*s} > 0$: $\lambda'^s = \lambda^{*s}$, the IC at (λ^*, ϕ^*, U) gives $U^s(k^s) - U^s(i) \leq \lambda^{*s} c_i$, and continuity of LHS in U'' preserves the inequality with small slack for U'' close to U (the RHS doesn't change). When $\lambda^{*s} = 0$: $\lambda'^s = \epsilon$, and the IC reads $U''^s(k^s) - U''^s(i) \leq \epsilon c_i$. As $U'' \to U$, LHS $\to U^s(k^s) - U^s(i) = 0 - U^s(i) \leq 0$ (since $U^s(i) \geq 0$), so for U'' close enough, LHS $\leq \epsilon c_i/2 < \epsilon c_i$.

Convergence of values. On sides with $\lambda^{*s} > 0$: $\lambda'^s = \lambda^{*s}$ and $\phi'(U)^s = \phi^{*s}$, so those terms are unchanged. On sides with $\lambda^{*s} = 0$: $\phi'(U)^s = \phi^{*s} = u^s(k^s, k^{\bar{s}})$ (the extremal fee; ϕ^{*s} is free in its range at $\lambda^{*s} = 0$, and we may assume the selected maximizer places ϕ^{*s} there as it maximizes $\phi^a + \phi^b$). As $\epsilon \to 0$, $m(\lambda') \to m(\lambda^*)$ by continuity. The value $m(\lambda')(\phi'(U)^a + \phi'(U)^b) \to m(\lambda^*)(\phi^{*a} + \phi^{*b})$. ∎

Proof of Lemma 11. Upper semicontinuity of each $\hat{V}(k^a, k^b; \cdot)$ on $\mathcal{U}$ is Lemma 12, as is nonemptiness and closed graph of the argmax. Upper semicontinuity of the finite max V^*

follows from USC of each $\hat{V}(k^a, k^b; \cdot)$.

For lower semicontinuity of V^* on $\mathcal{U}$: let $U_n \to U$ in $\mathcal{U}$. If $V^*(U) = 0$, trivial. Otherwise pick $(k^a, k^b) \in \arg\max \hat{V}(\cdot; U)$ with $\hat{V}(k^a, k^b; U) = V^*(U) > 0$ and a maximizer $(\lambda^*, \phi^*) \in \arg\max$ (29); since $\hat{V}(k^a, k^b; U) > 0$ we have $m(\lambda^*) > 0$ and $\phi^{*a} + \phi^{*b} > 0$. Given $\delta > 0$, Lemma 13 supplies a perturbed feasible point $(\lambda', \phi'(\cdot))$ with value at least $V^*(U) - \delta$ (for ϵ small), along with a neighborhood N of U in $\mathcal{U}$ on which $(\lambda', \phi'(U'')) \in L(k^a, k^b; U'')$. For $U_n \in N$:

$$V^*(U_n) \;\geq\; \hat{V}(k^a, k^b; U_n) \;\geq\; m(\lambda')\big(\phi'(U_n)^a + \phi'(U_n)^b\big) \;\xrightarrow{n\to\infty}\; m(\lambda')\big(\phi'(U)^a + \phi'(U)^b\big) \;\geq\; V^*(U) - \delta.$$

Since $\delta > 0$ was arbitrary, $\liminf V^*(U_n) \geq V^*(U)$. Continuity of V^* on $\mathcal{U}$ follows. ∎

F.4 Proof of Theorem 1

Proof of Theorem 1. Define the argmax set of markets achieving free entry,

$$A(U) \;\equiv\; \{(k^a, k^b) : L(k^a, k^b; U) \neq \emptyset \text{ and } \hat{V}(k^a, k^b; U) = c\}.$$

By Lemma 11, V^* is continuous and each $\hat{V}(k^a, k^b; \cdot)$ is upper semicontinuous on $\mathcal{U}$. Combining the two, $U \mapsto \arg\max_{(k^a,k^b)} \hat{V}(\cdot; U)$ is upper hemicontinuous on $\mathcal{U}$: if $(k^a, k^b) \in \arg\max(U_n)$ with $U_n \to U$, then $\hat{V}(k^a, k^b; U_n) = V^*(U_n)$; taking $\limsup$, upper semicontinuity gives $\hat{V}(k^a, k^b; U) \geq V^*(U)$, and $\hat{V} \leq V^*$ gives equality. The same argument with c in place of V^* gives upper hemicontinuity of A on $\mathcal{U}$.

Platform and auctioneer correspondences Throughout, μ denotes the match measure of Definition 2. Side-s agent flows are recovered from μ by the Consistency condition: at a separating market (k^a, k^b) with matching probability $\lambda^{s*} > 0$, agent flow on side s is $\mu(k^a, k^b)/\lambda^{s*}$. Match measure is uniformly bounded by total population, which makes the fixed-point argument tractable.

Pick $\bar{\mu} > \sum_{s,i} I^s F^s(\{i\})$ strictly exceeding total population on both sides combined—any market-clearing allocation has $\sum \mu \leq \min_s I^s$, so this upper bound is never binding at a CSE—and set $\mathcal{M} \equiv [0, \bar{\mu}]^{|\square^a| \times |\square^b|}$.

For each (k^a, k^b) with $L(k^a, k^b; U) \neq \emptyset$, Lemma 11 gives that $\arg\max$ (29) is nonempty, compact, and has closed graph in U. The Kuratowski–Ryll-Nardzewski measurable selection theorem (Kuratowski and Ryll-Nardzewski, 1965) then supplies a measurable selection $(\lambda^*(k^a, k^b; \cdot), \phi^*(k^a, k^b; \cdot))$. We pick this selection to satisfy: $\lambda^{s*}(k^a, k^b; U) > 0$ *whenever* $U^s(k^s) > 0$. This property is automatic at every point of the argmax, because (P) requires

$\lambda^s(u^s - \phi^s) = U^s(k^s)$, which forces $\lambda^s > 0$ when $U^s(k^s) > 0$.[23] Combining (P) and (B), whenever $U^s(k^s) > 0$,

$$\lambda^{s*}(k^a, k^b; U) \;=\; \frac{U^s(k^s)}{u^s(k^s, k^{\bar{s}}) - \phi^{s*}} \;\geq\; \frac{U^s(k^s)}{u^s(k^s, k^{\bar{s}}) + u^{\bar{s}}(k^{\bar{s}}, k^s)} \;=\; \frac{U^s(k^s)}{f(k^s, k^{\bar{s}})},$$

so λ^{s*} is bounded away from zero on any subset of $\mathcal{U}$ where $U^s(k^s)$ is bounded away from zero.

The *platform correspondence* $\Gamma_\mu : \mathcal{U} \rightrightarrows \mathcal{M}$ is

$$\Gamma_\mu(U) = \begin{cases} \{\mu \in \mathcal{M} : \operatorname{supp}(\mu) \subseteq \arg\max_{(k^a,k^b)} \hat{V}(\cdot; U),\ \sum \mu = \bar{\mu}\} & V^*(U) > c, \\ \{\mu \in \mathcal{M} : \operatorname{supp}(\mu) \subseteq A(U)\} & V^*(U) = c, \\ \{0\} & V^*(U) < c. \end{cases}$$

Γ_μ is nonempty, compact- and convex-valued, and upper hemicontinuous: the three cases are locally mutually exclusive in U by continuity of V^*, and the support-containment constraint is closed by upper hemicontinuity of the argmax correspondence and of A.

For each (s, i), define *demand* and *excess demand*:

$$D^s(i; U, \mu) \equiv \sum_{\substack{(k^a,k^b):\, k^s = i \\ \lambda^{s*}(k^a,k^b;U) > 0}} \frac{\mu(k^a, k^b)}{\lambda^{s*}(k^a, k^b; U)}, \qquad Z^s(i; U, \mu) \equiv D^s(i; U, \mu) - I^s F^s(\{i\}).$$

This agrees with the side-s agent flow ν^s of Definition 2: at a separating market (k^a, k^b) with $k^s = i$, the Consistency condition $d\mu = \lambda^{s*}\, d\nu^s$ gives type-i flow $\mu(k^a, k^b)/\lambda^{s*}$, which is the summand in D^s. By the bound on λ^{s*} above, $D^s(i; U, \mu)$ is continuous in (U, μ) on $\{U : U^s(i) > 0\}$.

The *auctioneer correspondence* $\Gamma_U : \mathcal{U} \times \mathcal{M} \rightrightarrows \mathcal{U}$ is

$$\Gamma_U(U, \mu) \;=\; \arg\max_{U' \in \mathcal{U}} \sum_{s,i} U'^s(i)\, Z^s(i; U, \mu).$$

The auctioneer raises $U'^s(i)$ toward C when type i has excess demand and lowers it toward 0 when type i has excess supply, in the spirit of the Gale–Debreu–Nikaido lemma.

We now apply a Kakutani fixed-point argument. The joint correspondence $\Gamma(U, \mu) \equiv \Gamma_U(U, \mu) \times \Gamma_\mu(U)$ has nonempty, convex, compact values (linearity of the auctioneer's ob-

[23] When $U^s(k^s) = 0$, the argmax may admit both $\lambda^s = 0$ and, if the matching function satisfies $m(0^+, \lambda^{\bar{s}}) > m(\lambda^s, \lambda^{\bar{s}})$ strictly, only $\lambda^s = 0$ is optimal. The selection takes $\lambda^{s*} = 0$ there; nothing in the argument below depends on the tie-breaking choice.

jective and compactness of $\mathcal{U}$; convexity of the support-containment constraint in μ and compactness of $\mathcal{M}$). The remaining hypothesis is closed graph, which holds on the interior domain $\mathcal{U}^\circ \equiv \{U \in \mathcal{U} : U^s(i) > 0 \text{ for all } (s,i)\}$: on $\mathcal{U}^\circ \times \mathcal{M}$, the bound $\lambda^{s*} \geq U^s(k^s)/f$ gives a uniform positive lower bound, D^s is continuous, Z^s is continuous, and the linear auctioneer's objective inherits continuity, so Berge's theorem delivers closed graph of the argmax.

At boundary points where $U^s(i) = 0$ for some (s,i), the demand $D^s(i;\cdot,\cdot)$ can be discontinuous: along sequences $U_n^s(k^s) \downarrow 0$ with μ_n bounded away from zero at some market with $k^s = i$, the selected $\lambda_n^{s*} \downarrow 0$ drives μ_n/λ_n^{s*} unbounded, whereas the limit selection at $U^s(k^s) = 0$ has $\lambda^{s*} = 0$ and contributes nothing to $D^s(i;U,\mu)$. To handle the boundary, we use a standard perturbation: for $\eta \in (0, \min_{s,i} I^s F^s(\{i\})/2)$, let $\mathcal{U}_\eta \equiv [\eta, C]^{|\mathbb{I}^a|+|\mathbb{I}^b|}$. On $\mathcal{U}_\eta$, the lower bound $\lambda^{s*} \geq \eta/f$ is uniform, so D^s is continuous, Γ_U has closed graph by Berge, and Kakutani's theorem applied to Γ restricted to $\mathcal{U}_\eta \times \mathcal{M}$ delivers a fixed point (U_η^*, μ_η^*).[24]

Take a sequence $\eta_n \downarrow 0$. By compactness of $\mathcal{U} \times \mathcal{M}$, the sequence $(U_{\eta_n}^*, \mu_{\eta_n}^*)$ has a convergent subsequence; call the limit $(U^*, \mu^*) \in \mathcal{U} \times \mathcal{M}$. We will verify that (U^*, μ^*) satisfies the conditions needed to build a CSE.

Verification We show that the limit point (U^*, μ^*) satisfies conditions (i)–(iii) of Proposition 1 Part 2. Throughout, $(U_{\eta_n}^*, \mu_{\eta_n}^*) \to (U^*, \mu^*)$ along the chosen subsequence, and we abbreviate $\eta \equiv \eta_n$ when context is clear.

Step 1: $V^*(U^*) \leq c$. Suppose for contradiction $V^*(U^*) > c$. By continuity of V^* (Lemma 11), $V^*(U_\eta^*) > c$ for all sufficiently small η, so the platform correspondence puts $\sum \mu_\eta^* = \bar{\mu}$ on the tail. At any argmax market for such η, $V^*(U_\eta^*) > 0$ forces $m(\lambda^*) > 0$ and $\phi^{a*} + \phi^{b*} > 0$; combining with (P), at least one side s has $U_\eta^{*s}(k^s) > 0$, so the selection rule gives $\lambda^{s*} > 0$ on that side. Counting contributions to demand:

$$\sum_s \sum_i D^s(i; U_\eta^*, \mu_\eta^*) \;=\; \sum_s \sum_{(k^a,k^b):\lambda^{s*}>0} \frac{\mu_\eta^*(k^a,k^b)}{\lambda^{s*}} \;\geq\; \sum_{(k^a,k^b)} \mu_\eta^*(k^a,k^b) \;=\; \bar{\mu}, \tag{30}$$

where the inequality uses $\lambda^{s*} \leq 1$ on the contributing side and the fact that at least one side contributes at each market in the support. Since $\bar{\mu} > \sum_{s,i} I^s F^s(\{i\})$, this gives $\sum_{s,i} Z^s(i; U_\eta^*, \mu_\eta^*) > 0$, so $Z^s(i; U_\eta^*, \mu_\eta^*) > 0$ for some (s,i). The auctioneer on $\mathcal{U}_\eta$ with a positive coefficient on $U'^s(i)$ then sets $U_\eta^{*s}(i) = C$.

But $U_\eta^{*s}(i) = C$ is incompatible with feasibility at any market serving type i: from (P)

[24] On $\mathcal{U}_\eta$ the auctioneer maximizes over $[\eta, C]^{\cdot}$ rather than $[0, C]^{\cdot}$; at a fixed point, $U_\eta^{*s}(i) = \eta$ plays the role that $U^{*s}(i) = 0$ plays on $\mathcal{U}$, and the auctioneer's FOC reads $Z^s \leq 0$ at $U_\eta^{*s}(i) = \eta$, $Z^s = 0$ at interior, and $Z^s \geq 0$ at $U_\eta^{*s}(i) = C$.

and $U^{*s}_\eta(i) = C > 0$, we get $\phi^s = u^s(i, k^{\bar{s}}) - C/\lambda^s$ with $\lambda^s > 0$; substituting into (F′) or directly combining (F) with (B) on side $\bar{s}$,

$$0 \;\le\; \phi^s + \phi^{\bar{s}} \;\le\; \big(u^s(i, k^{\bar{s}}) - C/\lambda^s\big) + u^{\bar{s}}(k^{\bar{s}}, i) \;=\; f(i, k^{\bar{s}}) - C/\lambda^s,$$

so $C \le \lambda^s f(i, k^{\bar{s}}) \le f(i, k^{\bar{s}}) \le \max f$, contradicting the choice of $C > \max f$. So $L(k^a, k^b; U^*_\eta) = \emptyset$ at every market with $k^s = i$, hence such markets cannot lie in $\arg\max \hat{V}(\cdot; U^*_\eta)$, contradicting that μ^*_η places positive mass there. Thus $V^*(U^*_\eta) \le c$ for all sufficiently small η, and by continuity of V^*, $V^*(U^*) \le c$.

Step 2: Market Clearing. Fix (s, i) and consider the auctioneer's FOC at (U^*_η, μ^*_η) for small η: $Z^s(i; U^*_\eta, \mu^*_\eta) \le 0$ if $U^{*s}_\eta(i) = \eta$ (lower boundary), $Z^s = 0$ if $U^{*s}_\eta(i) \in (\eta, C)$, and $U^{*s}_\eta(i) = C$ is ruled out by Step 1 at any type with positive demand. Take the limit $\eta \downarrow 0$. If $U^{*s}(i) = 0$, then $U^{*s}_\eta(i) = \eta$ eventually, and $Z^s(i; U^*_\eta, \mu^*_\eta) \le 0$ passes to the limit because at each η, market clearing gives $\mu^*_\eta(k^a, k^b) \le \lambda^{s*}(k^a, k^b; U^*_\eta) \cdot I^s F^s(\{i\})$ at every market with $k^s = i$, and $\lambda^{s*}(k^a, k^b; U^*_\eta) \to 0$ along the sequence (by closed graph of $\arg\max$ (29) from Lemma 11 and the fact that at $U^{*s}(i) = 0$ the argmax has $\lambda^{s*} = 0$ on side s); hence $\mu^*(k^a, k^b) = 0$ at every market with $k^s = i$, so $D^s(i; U^*, \mu^*) = 0$ and $Z^s(i; U^*, \mu^*) = -I^s F^s(\{i\}) \le 0$.[25] If instead $U^{*s}(i) > 0$, then $U^{*s}_\eta(i) \in (\eta, C)$ eventually (since $\eta \downarrow 0$ while $U^{*s}_\eta(i) \to U^{*s}(i) > 0$), so $Z^s(i; U^*_\eta, \mu^*_\eta) = 0$ on the tail, and continuity of D^s on $\mathcal{U}^\circ$ gives $Z^s(i; U^*, \mu^*) = 0$. In either case, Market Clearing (Definition 2) holds: $\sum_{i \in \mathbb{I}'} Z^s(i; U^*, \mu^*) \le 0$ for any $\mathbb{I}' \subseteq \mathbb{I}^s$, with equality when $U^{*s}(i) > 0$ for all $i \in \mathbb{I}'$.

Step 3: Verifying a CSE. We split into two cases.

Case A: $V^*(U^*) = c$. On the tail, μ^*_η is supported on $A(U^*_\eta)$; upper hemicontinuity of A (established in the first paragraph of the proof) yields $\mathrm{supp}(\mu^*) \subseteq A(U^*)$, so every $(k^a, k^b) \in \mathrm{supp}(\mu^*)$ has $L(k^a, k^b; U^*) \neq \emptyset$ and $\hat{V}(k^a, k^b; U^*) = c$. Since $c > 0$, this forces $m(\lambda^*) > 0$ and $\phi^{a*} + \phi^{b*} > 0$ at every such market. Every $\tau \in \mathrm{supp}(\mu^*)$ is a separating terms-of-trade at (k^a, k^b) with $(\lambda^{a*}, \lambda^{b*}, \phi^{a*}, \phi^{b*}) \in \arg\max$ (29) and value c, so condition (i) of Proposition 1 Part 2 holds. Step 2 gives Market Clearing.

For Equilibrium Utility, we must show $U^{*s}(i) \ge \bar{U}^s(i, \tau, \lambda^{s*})$ for every i and $\tau \in \mathrm{supp}(\mu^*)$. When τ serves $k^s = i$, equality follows from (P). When $k^s > i$, the downward constraint (IC) enforced by $\arg\max$ (29) gives the bound. When $k^s < i$, the bound is an *upward* incentive constraint; we rule out its violation as follows. Suppose $\bar{U}^s(i, \tau, \lambda^{s*}) > U^{*s}(i)$ at some $\tau \in \mathrm{supp}(\mu^*)$ with $k^s < i$. The construction in the proof of Corollary 1—which uses only

[25] The vanishing of μ^*_η at blow-up markets is what prevents the discontinuity of D^s at the boundary from affecting the limit: although μ/λ^{s*} is individually unbounded as $U^s \to 0$, market clearing at each η forces μ^*_η to shrink at the same rate as λ^{s*}, keeping the ratio bounded by supply $I^s F^s(\{i\})$.

Common Ranking, Supermodularity, and the existence of τ itself, and does *not* assume that (U^*, μ^*) is already a CSE—produces a separating terms-of-trade $\tilde{\tau}$ targeting a type $\tilde{k}^s \geq i$ on side s, feasible at U^*, with value strictly exceeding $m(\lambda^*(\tau))(\phi^{a*} + \phi^{b*}) = c$. This contradicts $V^*(U^*) \leq c$ (Step 1), so no such τ exists. Hence condition (ii) holds. Condition (iii) is $V^*(U^*) = c$ by the case assumption. Applying Proposition 1 Part 2, (U^*, μ^*) is a CSE.

Case B: $V^*(U^*) < c$. On the tail, $\mu^*_\eta = 0$ and $Z^s(i; U^*_\eta, 0) = -I^s F^s(\{i\}) \leq 0$, so the auctioneer's optimum on $\mathcal{U}_\eta$ places $U^{*s}_\eta(i) = \eta$ for every (s, i). Hence $U^* = 0$ and $\mu^* = 0$. Take $T = \emptyset$, $\mu^* = 0$, and Λ assigning any $\lambda \in \mathbb{\Lambda}$ on T^p; Free Entry is vacuous, Market Clearing gives $0 \leq$ supply with no positive-utility types to enforce equality, and the Equilibrium Utility condition gives $U^{*s}(i) = \max\{0, \sup_{\tau \in T} \bar{U}^s(i, \tau, \Lambda^s(\tau))\} = 0$, reading the sup over the empty set as $-\infty$. The trivial CSE $(U^* \equiv 0, \mu^* = 0)$ is verified directly from Definition 2. ■

F.5 Non-existence under Strict DWTP

Under strict DWTP, the fixed-point argument underlying Theorem 1 fails. This subsection exhibits a two-type, symmetric example satisfying Common Ranking, Supermodularity, and strict DWTP in which no CSE exists for a range of values of c. This example shows that the theorem's restriction to strict IWTP is substantive, not merely a technical convenience.

Types: $\mathbb{I}^a = \mathbb{I}^b = \{0, 1\}$, symmetric payoffs: $u^a(i, j) = u^b(i, j) \equiv u(i, j)$ with

$$u(0,0) = 3, \quad u(0,1) = 5, \quad u(1,0) = 1, \quad u(1,1) = 4.$$

Matching function: $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ on the unit simplex $\mathbb{\Lambda} = \{(\lambda^a, \lambda^b) \geq 0 : \lambda^a + \lambda^b \leq 1\}$, populations: $F^a(\{i\}) = F^b(\{i\}) = 1/2$ with $I^a = I^b = 1$, intermediation cost: $c = 7$. We can verify (strict) Common Ranking, (strict) Supermodularity and strict DWTP directly.

We claim no CSE exists in this example. By Proposition 1, any CSE is separating. Additionally, all feasible terms of trade have $\phi^s \leq u(k^s, k^{\bar{s}})$. Thus a $(0,0)$, $(0,1)$, and $(1,0)$ match all have $\phi^a + \phi^b \leq 6$. Since $m \leq 1$, it follows that $c > m(\lambda^a, \lambda^b)(\phi^a + \phi^b)$ in any terms-of-trade that attracts type 0 agents. Since no such terms of trade can exist, $U^a(0) = U^b(0) = 0$. Since U^s is nonincreasing (Lemma 2), $U^a(1) = U^b(1) = 0$ as well.

Now consider a $(1,1)$ market with $\phi^a = \phi^b = 4$. The incentive and participation constraints are satisfied if and only if $\lambda^s = 0$, giving a platform matching probability of 1 and hence profits of $\phi^a + \phi^b = 8 > 7$. Thus this deviation is profitable, so no CSE exists.

The applications in Section 6 and Section 7.2 of the main text construct DWTP equilibria directly without relying on Theorem 1; the present example does not affect those results.

F.6 Non-existence under Non-Monotone WTP

Monotone WTP cannot be dispensed with in Theorem 1. This subsection exhibits a three-type economy in which Common Ranking and Supermodularity hold, but Monotone WTP fails, and no symmetric CSE exists. The failure traces to the conflicting IC bounds in Step 4: with $u(\cdot, 3)$ first increasing then decreasing across types, the downward ICs from types 1 and 2 onto the $(3, 3)$ market act in opposite directions, and $\hat{V}(3, 3; U)$ is discontinuous as the two bounds cross. This is exactly the mechanism that Monotone WTP is designed to preclude in Lemma 11.

Set $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ on the unit simplex ($\lambda^a, \lambda^b \geq 0$, $\lambda^a + \lambda^b \leq 1$), the parametric matching function of the main text with $\gamma = 1$. Intermediation costs are $c = 1$. Types are labelled $1, 2, 3$; populations and payoffs are symmetric across sides, so $F^a = F^b \equiv F$ and $u^a(i, j) = u^b(j, i) \equiv u(i, j)$. Match payoffs are

$$\big[u(i, j)\big]_{i,j=1,2,3} = \begin{pmatrix} 1.000 & 1.001 & 1.090 \\ 1.090 & 1.100 & 1.210 \\ 0.010 & 0.100 & 1.200 \end{pmatrix}.$$

One can verify by direct computation that this payoff matrix satisfies Common Ranking (column-wise increasing) and strict Supermodularity. It does *not* satisfy Monotone WTP: reading down a column, $u(i, j)$ first increases from $i = 1$ to $i = 2$, then decreases sharply from $i = 2$ to $i = 3$.

By Proposition 1, any CSE is separating, with active markets drawn from the six type pairs $(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)$. We look for a symmetric equilibrium in which $U^a(i) = U^b(i) \equiv U(i)$ and show no such U exists.

Step 1: Lower bounds on $U(1)$ and $U(2)$. Consider an unconstrained $(1, 1)$ market: the value $\hat{V}_1(U(1)) \equiv \max_{(\lambda^a,\lambda^b)\in\Lambda}(1 - \lambda^a - \lambda^b)\big(2u(1, 1) - U(1)/\lambda^a - U(1)/\lambda^b\big)$ is strictly decreasing in $U(1)$. Defining $U(1)^*$ by $\hat{V}_1(U(1)^*) = c = 1$ gives $U(1)^* = \frac{3}{4} - 1/\sqrt{2} \approx 0.0429$. If $U(1) < U(1)^*$ the $(1, 1)$ market earns supernormal profit, inconsistent with CSE; hence $U(1) \geq U(1)^*$.

Similarly, define $U(2)^*$ by setting the unconstrained $(2, 2)$-market value to c; direct computation gives $U(2)^* = \frac{1}{10}(8 - \sqrt{55}) \approx 0.0584$. One verifies that at $U(2)^*$ and any $U(1) \geq U(1)^*$, the downward-incentive constraint $U(1) - U(2)^* \geq \lambda^s(u(1, 2) - u(2, 2))$ is slack at the unconstrained maximizer; so $(2, 2)$-market profit reaches c at $U(2)^*$ and exceeds c at any $U(2) < U(2)^*$, giving $U(2) \geq U(2)^*$.

Step 2: Markets $(1,2)$, $(1,3)$, and $(2,3)$ cannot be active. The unconstrained $(1,2)$-market value at $(U(1)^*, U(2)^*)$ equals

$$\max_{(\lambda^a,\lambda^b)\in\Lambda}(1-\lambda^a-\lambda^b)\big(u(1,2)+u(2,1)-U(1)^*/\lambda^a-U(2)^*/\lambda^b\big)\approx 0.9946<c.$$

Since the constrained value is weakly lower, and raising $U(1)$ or $U(2)$ further only decreases it, no $(1,2)$ market can be active. Analogous calculations give unconstrained values for $(1,3)$ and $(2,3)$ at $U(3)=0$ of ≈ 0.7085 and ≈ 0.8153 respectively, both below c. Markets $(1,3)$ and $(2,3)$ therefore cannot be active either.

Step 3: Market clearing forces $(1,1),(2,2),(3,3)$ all active. Given that $(1,2),(1,3),(2,3)$ are inactive, type i on either side must match only at the (i,i) market. Market Clearing then requires all three of $(1,1),(2,2),(3,3)$ to be active. Free entry pins $U(1)=U(1)^*$, $U(2)=U(2)^*$.

Step 4: The $(3,3)$ market cannot clear at profit c. At $(U(1)^*, U(2)^*)$ and any $U(3)\geq 0$, the $(3,3)$-market value solves

$$\max_{(\lambda^a,\lambda^b)\in\Lambda}(1-\lambda^a-\lambda^b)\big(2u(3,3)-U(3)/\lambda^a-U(3)/\lambda^b\big)$$

subject to the two downward incentive constraints

$$U(1)^*-U(3)\geq\lambda^s\big(u(1,3)-u(3,3)\big),$$
$$U(2)^*-U(3)\geq\lambda^s\big(u(2,3)-u(3,3)\big).$$

This is where the failure of Monotone WTP bites: the two coefficients on λ^s have *opposite* signs. $u(1,3)-u(3,3)=-0.11<0$, so the first constraint is a lower bound on λ^s; $u(2,3)-u(3,3)=0.01>0$, so the second is an upper bound. Every $U(3)$ above the value 0.0571 at which both constraints bind simultaneously violates one or the other.

At $U(3)=0.0571$, both constraints bind simultaneously at $\lambda^a=\lambda^b\approx 0.1291$ and the $(3,3)$-market value equals $\approx 1.1242>c$. The market value is strictly decreasing in $U(3)$ (through both the objective and the tightening of the constraints), so at any smaller $U(3)$ the value is at least 1.1242. Thus the $(3,3)$ market earns supernormal profit whenever $(1,1)$ and $(2,2)$ break even, contradicting free entry. This proves no symmetric CSE exists.

Remark on asymmetric CSE. The construction above rules out symmetric CSE. The same mechanism — conflicting IC coefficients at the $(3,3)$ market — obstructs asymmetric

CSE as well, as we have verified by numerical grid search over the CSE-consistent region of (U^a, U^b). A rigorous analytical proof requires tracking which incentive constraints bind at the $(1, 2)$, $(2, 1)$, and $(2, 2)$ markets across the four-dimensional parameter space of type-1 and type-2 utilities, which we find tangential to the purpose of this example. The qualitative lesson — that Monotone WTP cannot be dropped from Theorem 1 — is already established by Steps 1–4.

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Guide to model configurations

Configuration	Willingness-to-pay	Cost	Symmetry	Location
PAM	IWTP	Zero	Symmetry	Section 5
Anchor	IWTP	Zero	Symmetry	Section 5
PAM	DWTP	Zero	Symmetry	Section 6
Anchor	DWTP	Zero	Symmetry	Section 6
PAM	IWTP/DWTP	Zero	Asymmetry	Section 7.1
PAM	IWTP/DWTP	Positive	Symmetry	Section 7.2
NAM	IWTP/DWTP	Zero	Symmetry	Online Appendix G
PAM	IWTP/DWTP	Positive	Asymmetry	Online Appendix H

G Negative Sorting with Zero Costs and Symmetry

This section characterizes negative assortative matching (NAM) CSE in a symmetric environment with zero intermediation costs, for both IWTP and DWTP. Characterization 2 gives the differential equation system for the IWTP case, and Characterization 3 does the same for DWTP. We illustrate the IWTP case with a numerical example in Figure 8, where we also verify global incentive constraints and the absence of profitable platform deviations.

Characterization 2 (NAM with Symmetry and Zero Costs under IWTP) *Assume Common Ranking, Supermodularity, Limit Supermodularity, Symmetric Environment, and IWTP. Also assume $c = 0$. In a negative sorted equilibrium, for the terms-of-trade attracting types $(i, \sigma(i))$, denote the side-a and side-b matching probabilities as $\ell^a(i)$ and $\ell^b(i)$. The following equation system characterizes the equilibrium outcomes:*

$$m(\ell^a(i), \ell^b(i)) = 0,$$

$$U'(i) = \ell^a(i)\, u_1(i, \sigma(i)), \qquad U'(\sigma(i)) = \ell^b(i)\, u_1(\sigma(i), i),$$

$$U(i) = \ell^a(i)\, (u(i, \sigma(i)) - \Phi^a(i)), \qquad U(\sigma(i)) = \ell^b(i)\, (u(\sigma(i), i) - \Phi^b(i)),$$

$$\frac{\ell^{a\prime}(i)}{\ell^a(i)} + \frac{u_{12}(i, \sigma(i))}{u_1(i, \sigma(i))}\sigma'(i) = \frac{m_2(\ell^a(i), \ell^b(i))\, \ell^b(i)}{m_1(\ell^a(i), \ell^b(i))\, \ell^a(i)}\, \frac{u_{12}(\sigma(i), i)}{u_1(\sigma(i), i)},$$

$$\sigma'(i) = -\frac{f(i)\, \ell^a(i)}{f(\sigma(i))\, \ell^b(i)},$$

with boundary conditions $\sigma(\underline{i}) = \bar{i}$, $\sigma(\bar{i}) = \underline{i}$, and $\Phi^a(\underline{i}) + \Phi^b(\bar{i}) = 0$ in the $(\underline{i}, \bar{i})$ market.

The system parallels Characterization 1 for PAM with zero costs, with two differences. First, σ is decreasing, so σ' has a negative sign. Second, the boundary conditions pair the lowest side-a type with the highest side-b type (and vice versa). As in the PAM case, the zero-fee condition at the $(\underline{i}, \bar{i})$ market pins down the level of U; after this computation, one still has to verify global incentive constraints and the absence of profitable platform deviations, which we do numerically in the example below.

Figure 8 illustrates the equilibrium and compares it to the observable-types benchmark. Many patterns are similar across the two environments: higher types match with lower types, higher types match with a higher probability, and lower types pay a higher fee. The platform matching probability is zero throughout. The main difference is that the sum of fees collected is zero with observable types but strictly positive with private information, except in the $(\underline{i}, \bar{i})$ market.

Proof of Characterization 2. The derivation parallels the proof of Characterization 1; we indicate the points of difference.

Zero platform matching probability and local IC. Lemma 4 gives $m(\ell^a(i), \ell^b(i)) = 0$, the first equation. For a platform attracting (i, j), the local ICs give $\lambda^a(i, j) = U'(i)/u_1(i, j)$ and $\lambda^b(i, j) = U'(j)/u_1(j, i)$, the second line. The third line is the participation constraint in this same market.

Platform optimality. Writing $\hat{V}(i, j) = m(\lambda^a(i, j), \lambda^b(i, j)) \cdot S(i, j)$ with S the sum of fees, and using $m(\ell^a(i), \ell^b(i)) = 0$ as before, the FOC $\hat{V}_1(i, \sigma(i)) = 0$ reduces to

$$0 = m_1(\ell^a(i), \ell^b(i)) \left.\frac{\partial \lambda^a}{\partial i}\right|_{j=\sigma(i)} + m_2(\ell^a(i), \ell^b(i)) \left.\frac{\partial \lambda^b}{\partial i}\right|_{j=\sigma(i)}. \tag{31}$$

The same calculations as in the proof of Characterization 1 give

$$\left.\frac{\partial \lambda^b}{\partial i}\right|_{j=\sigma(i)} = -\ell^b(i) \frac{u_{12}(\sigma(i), i)}{u_1(\sigma(i), i)}, \qquad \left.\frac{\partial \lambda^a}{\partial i}\right|_{j=\sigma(i)} = \ell^{a\prime}(i) + \ell^a(i) \frac{u_{12}(i, \sigma(i))}{u_1(i, \sigma(i))} \sigma'(i).$$

Substituting into (31) and dividing by $m_1(\ell^a(i), \ell^b(i))\, \ell^a(i)$ yields the fourth line in the characterization.

Market clearing. In NAM, side-a type i matches with side-b type $\sigma(i)$, where σ is decreasing. Consistency of the equilibrium measures gives $d\nu^b/d\nu^a = \ell^a/\ell^b$ on the active curve. Equating accumulated side-b agents used by side-a types in $[\underline{i}, i]$ with available supply in $[\sigma(i), \bar{i}]$:

$$I^b\big(F(\bar{i}) - F(\sigma(i))\big) = \int_{\underline{i}}^{i} \frac{\ell^a(x)}{\ell^b(x)}\, I^a\, dF(x).$$

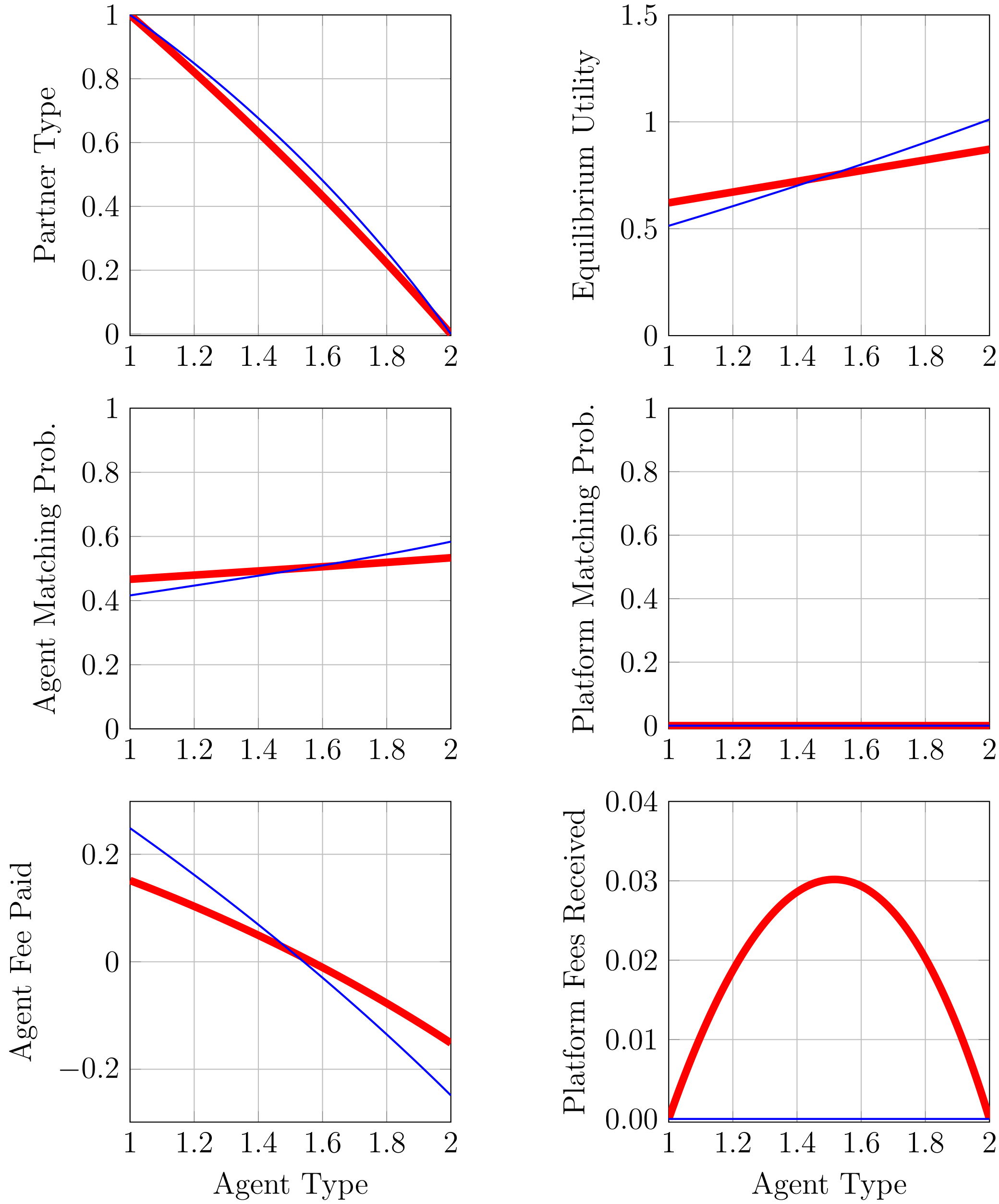

Figure 8: Negative Assortative Matching with IWTP. Notes: Thick red lines represent private-information equilibrium outcomes, while thinner blue lines show observable-type equilibrium outcomes. Payoff function $u(i,j) = (0.5i^{0.8} + 0.5j^{0.8})^{1.25}$ $(\theta = 5)$, matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ $(\gamma = 1)$, types are distributed uniformly on $[1, 2]$.

Under Symmetric Environment, $I^a = I^b$ and the two sides share the distribution F. Differentiating and solving:

$$\sigma'(i) = -\frac{f(i)\,\ell^a(i)}{f(\sigma(i))\,\ell^b(i)} < 0,$$

the fifth line in the characterization.

Boundary conditions. Market clearing at the endpoints requires $\sigma(\underline{i}) = \bar{i}$ and $\sigma(\bar{i}) = \underline{i}$. In the $(\underline{i}, \bar{i})$ market, type $\underline{i}$ on side a faces no downward ICs, so λ^a is unconstrained, while λ^b must satisfy the local IC for type $\bar{i}$. The zero-fee condition $\Phi^a(\underline{i}) + \Phi^b(\bar{i}) = 0$ pins down the level of U. ■

Characterization 3 (NAM with Symmetry and Zero Costs under DWTP) *Assume Common Ranking, Supermodularity, Limit Supermodularity, Symmetric Environment, and DWTP. Also assume $c = 0$. In a negative sorted equilibrium, for the terms-of-trade attracting types $(i, \sigma(i))$, denote the side-a and side-b matching probabilities as $\ell^a(i)$ and $\ell^b(i)$. The following equation system characterizes the equilibrium outcomes:*

$$\begin{gathered}
u(i,\sigma(i)) + u(\sigma(i),i) - \frac{U(i)}{\ell^a(i)} - \frac{U(\sigma(i))}{\ell^b(i)} = 0,\\
U'(i) = \ell^a(i)\,u_1(i,\sigma(i)), \qquad U'(\sigma(i)) = \ell^b(i)\,u_1(\sigma(i),i),\\
U(i) = \ell^a(i)\,(u(i,\sigma(i)) - \Phi^a(i)), \qquad U(\sigma(i)) = \ell^b(i)\,(u(\sigma(i),i) - \Phi^b(i)),\\
u_2(\sigma(i),i) + \frac{U(i)}{\ell^a(i)}\left(\frac{\ell^{a\prime}(i)}{\ell^a(i)} + \frac{u_{12}(i,\sigma(i))}{u_1(i,\sigma(i))}\sigma'(i)\right) - \frac{U(\sigma(i))}{\ell^b(i)}\,\frac{u_{12}(\sigma(i),i)}{u_1(\sigma(i),i)} = 0,\\
\sigma'(i) = -\frac{f(i)\,\ell^a(i)}{f(\sigma(i))\,\ell^b(i)},
\end{gathered}$$

with boundary conditions $\sigma(\underline{i}) = \bar{i}$, $\sigma(\bar{i}) = \underline{i}$, and $m(\ell^a(\underline{i}), \ell^b(\underline{i})) = 0$ in the $(\underline{i}, \bar{i})$ market.

The DWTP case differs structurally from the IWTP case: with DWTP the platform matching probability is generically positive, while the sum of fees is zero in every active market. The first equation replaces $m = 0$ (which held under IWTP) with $\phi^a + \phi^b = 0$. The remaining equations are the local ICs, participation constraints, platform's partner-choice FOC, and market clearing, derived as in the IWTP case.

Proof of Characterization 3. The local ICs, participation constraints, and market-clearing derivation are identical to those in Characterization 2. With DWTP the platform matching probability is positive; instead the sum of fees is zero, i.e., $S(i, \sigma(i)) = 0$, where S is the fee sum from equation (10). This gives the first equation in the characterization.

Writing $\hat{V} = m \cdot S$, the FOC $\hat{V}_1(i, \sigma(i)) = 0$ becomes $m(\ell^a(i), \ell^b(i)) \cdot S_1(i, \sigma(i)) = 0$ (the $m_1 S$ and $m_2 S$ terms vanish because $S = 0$). Assuming $m(\ell^a(i), \ell^b(i)) \neq 0$, we require $S_1(i, \sigma(i)) = 0$. Differentiating S and using the local IC to cancel the $u_1(i, \sigma(i))$ term,

$$S_1(i, \sigma(i)) = u_2(\sigma(i), i) + \frac{U(i)}{(\ell^a(i))^2} \frac{\partial \lambda^a}{\partial i}\bigg|_{j=\sigma(i)} + \frac{U(\sigma(i))}{(\ell^b(i))^2} \frac{\partial \lambda^b}{\partial i}\bigg|_{j=\sigma(i)}.$$

Substituting the expressions for $\partial \lambda^a / \partial i$ and $\partial \lambda^b / \partial i$ from the proof of Characterization 2 yields the fourth equation in the characterization.

Finally, the boundary condition $m(\ell^a(\underline{i}), \ell^b(\underline{i})) = 0$ at the $(\underline{i}, \bar{i})$ market arises from the unconstrained problem for type $\underline{i}$ on side a. This pushes λ^a to the boundary of the feasible set, where $m = 0$. ■

H Equilibrium with Positive Costs and Asymmetry

This section characterizes CSE with positive costs and asymmetry between the two sides, generalizing the results in Section 7.1 (which assumed zero costs) and in Section 7.2 (which assumed symmetry). Section H.1 treats PAM, Section H.2 treats NAM. As in Section 7.2, the differential equations characterize a CSE whenever one exists; verifying existence requires checking the platform value $\hat{V}(k^a, k^b)$ in equation (10) for all (k^a, k^b), which is a numerical exercise.

H.1 Positive Assortative Matching

For expositional simplicity, we focus on equilibria where every type participates; *Discussion* at the end of this subsection addresses the possibility of non-participation. We denote by $[\underline{i}^s, \bar{i}^s]$ the support of the type distribution on side s, with density f^s.

Characterization 4 (PAM with Asymmetry and Positive Costs) *Assume Common Ranking and Supermodularity. In a PAM CSE with all types participating, for the terms-of-trade attracting types $(i, \sigma(i))$, denote the side-s matching probability as $\ell^s(i)$ and equilibrium* *utility as U^s. The following equation system characterizes the equilibrium outcomes:*

$$\xi^a(i)\left(\frac{\ell^{a\prime}(i)}{\ell^a(i)}+\frac{u^a_{12}(i,\sigma(i))}{u^a_1(i,\sigma(i))}\sigma'(i)\right)-\xi^b(i)\frac{u^b_{12}(\sigma(i),i)}{u^b_1(\sigma(i),i)}+u^b_2(\sigma(i),i)=0,$$
$$\xi^b(i)\frac{1}{\sigma'(i)}\left(\frac{\ell^{b\prime}(i)}{\ell^b(i)}+\frac{u^b_{12}(\sigma(i),i)}{u^b_1(\sigma(i),i)}\right)-\xi^a(i)\frac{u^a_{12}(i,\sigma(i))}{u^a_1(i,\sigma(i))}+u^a_2(i,\sigma(i))=0,$$
$$U^{a\prime}(i)=\ell^a(i)\,u^a_1(i,\sigma(i)),\qquad U^{b\prime}(\sigma(i))=\ell^b(i)\,u^b_1(\sigma(i),i),$$
$$U^a(i)=\ell^a(i)\,(u^a(i,\sigma(i))-\Phi^a(i)),\qquad U^b(\sigma(i))=\ell^b(i)\,(u^b(\sigma(i),i)-\Phi^b(i)),$$
$$\sigma'(i)=\frac{I^a f^a(i)\,\ell^a(i)}{I^b f^b(\sigma(i))\,\ell^b(i)},$$

with boundary conditions $\sigma(\underline{i}^a)=\underline{i}^b$ and $\sigma(\bar{i}^a)=\bar{i}^b$, together with the condition that, for any fixed $U^a(\underline{i}^a)$, the tuple $(\ell^a(\underline{i}^a),\ell^b(\underline{i}^a),U^b(\underline{i}^b))$ solves the unconstrained problem for the $(\underline{i}^a,\underline{i}^b)$ market:

$$c=\max_{\lambda^a,\lambda^b} m(\lambda^a,\lambda^b)\left(u^a(\underline{i}^a,\underline{i}^b)+u^b(\underline{i}^b,\underline{i}^a)-\frac{U^a(\underline{i}^a)}{\lambda^a}-\frac{U^b(\underline{i}^b)}{\lambda^b}\right). \tag{32}$$

Here

$$\xi^s(i)\equiv\frac{\epsilon_s(\ell^a(i),\ell^b(i))}{m(\ell^a(i),\ell^b(i))}c+\frac{U^s(k^s)}{\ell^s(i)},\qquad \epsilon_s(\lambda^a,\lambda^b)\equiv\frac{\lambda^s\,m_s(\lambda^a,\lambda^b)}{m(\lambda^a,\lambda^b)},$$

with $k^a=i$, $k^b=\sigma(i)$.

Proof of Characterization 4. The derivation parallels the proof of Characterization 1, with two main differences. First, with $c>0$ the platform matching probability m is generically nonzero, so the FOC for platform optimality picks up a term from $m\cdot S_1$ that was absent in the zero-cost case. Second, with asymmetry we obtain two FOCs (one from varying each side's type), yielding the first two lines in the characterization.

Local IC and market clearing. The local IC (Lemma 3) gives $\lambda^a(i,j)=U^{a\prime}(i)/u^a_1(i,j)$, $\lambda^b(i,j)=U^{b\prime}(j)/u^b_1(j,i)$, which is the third line. Market clearing gives the fifth line exactly as in the proof of Characterization 1. The fourth line is the participation constraint.

Platform optimality. Write $\hat{V}(i,j)=m(\lambda^a(i,j),\lambda^b(i,j))\cdot S(i,j)$, with $S(i,j)\equiv u^a(i,j)+u^b(j,i)-U^a(i)/\lambda^a(i,j)-U^b(j)/\lambda^b(i,j)$. Along the equilibrium path, $m\cdot S=c$, so $S=c/m$. Platform optimality gives $\hat{V}_1(i,\sigma(i))=0$. Applying the product rule:

$$\hat{V}_1=\left(m_1\frac{\partial\lambda^a}{\partial i}+m_2\frac{\partial\lambda^b}{\partial i}\right)S+m\,S_1.$$

Using the local IC to cancel the u_1^a term in S_1,

$$S_1\Big|_{j=\sigma(i)} = u_2^b(\sigma(i), i) + \frac{U^a(i)}{(\ell^a(i))^2}\frac{\partial \lambda^a}{\partial i}\bigg|_{j=\sigma(i)} + \frac{U^b(\sigma(i))}{(\ell^b(i))^2}\frac{\partial \lambda^b}{\partial i}\bigg|_{j=\sigma(i)}.$$

Substituting $S = c/m$ and collecting terms yields

$$\hat{V}_1 = \left(\frac{m_1\, c}{m} + \frac{m\, U^a(i)}{(\ell^a(i))^2}\right)\frac{\partial \lambda^a}{\partial i} + \left(\frac{m_2\, c}{m} + \frac{m\, U^b(\sigma(i))}{(\ell^b(i))^2}\right)\frac{\partial \lambda^b}{\partial i} + m\, u_2^b(\sigma(i), i).$$

Using the definitions of ξ^s and ϵ_s, each bracketed coefficient equals $m\,\xi^s(i)/\ell^s(i)$. Dividing through by m:

$$0 = \frac{\xi^a(i)}{\ell^a(i)}\frac{\partial \lambda^a}{\partial i}\bigg|_{j=\sigma(i)} + \frac{\xi^b(i)}{\ell^b(i)}\frac{\partial \lambda^b}{\partial i}\bigg|_{j=\sigma(i)} + u_2^b(\sigma(i), i).$$

The partial derivatives $\partial\lambda^a/\partial i$ and $\partial\lambda^b/\partial i$, evaluated at $j = \sigma(i)$, are the same as in the proof of Characterization 1:

$$\frac{\partial \lambda^a}{\partial i}\bigg|_{j=\sigma(i)} = \ell^{a\prime}(i) + \ell^a(i)\frac{u_{12}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))}\sigma'(i), \qquad \frac{\partial \lambda^b}{\partial i}\bigg|_{j=\sigma(i)} = -\ell^b(i)\frac{u_{12}^b(\sigma(i), i)}{u_1^b(\sigma(i), i)}.$$

Substituting yields the first line of the characterization.

The analogous FOC $\hat{V}_2(i, \sigma(i)) = 0$, obtained by varying j with i fixed, yields the second line by symmetric computations, using

$$\frac{\partial \lambda^a}{\partial j}\bigg|_{j=\sigma(i)} = -\ell^a(i)\frac{u_{12}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))}, \qquad \frac{\partial \lambda^b}{\partial j}\bigg|_{j=\sigma(i)} = \frac{1}{\sigma'(i)}\left(\ell^{b\prime}(i) + \ell^b(i)\frac{u_{12}^b(\sigma(i), i)}{u_1^b(\sigma(i), i)}\right),$$

where the second expression is derived by totally differentiating the identity $\ell^b(i) = \lambda^b(i, \sigma(i))$.

Boundary conditions. Market clearing at the endpoints requires $\sigma(\underline{i}^a) = \underline{i}^b$ and $\sigma(\bar{i}^a) = \bar{i}^b$. The $(\underline{i}^a, \underline{i}^b)$ market is unconstrained (Proposition 1), giving problem (32). ■

Discussion. If the differential equation system in Characterization 4 has no solution with $U^s(i) \geq 0$ for all i, then the PAM equilibrium involves non-participation by some types. Under IWTP, three scenarios are possible: low types on side a do not participate (there exists $i_*^a \in (\underline{i}^a, \bar{i}^a]$ with $U^a(i) = 0$ for $i \leq i_*^a$, and the assignment function starts from i_*^a); the analogous situation holds on side b; or the lowest types on both sides do not participate, which occurs when no terms-of-trade attracting side-s type-$\underline{i}^s$ covers the intermediation cost.

H.2 Negative Assortative Matching with Positive Costs

Our final example is a NAM CSE with positive costs. Its characterization is close to that of a PAM CSE.

Characterization 5 (NAM with Asymmetry and Positive Costs) *Assume Common Ranking and Supermodularity. In a NAM CSE with all types participating, for the terms-of-trade attracting types* $(i, \sigma(i))$*, denote the side-s matching probability as* $\ell^s(i)$ *and equilibrium utility as* U^s*. The equation system in the first four lines of Characterization 4 characterizes the equilibrium outcomes, together with*

$$\sigma'(i) = -\frac{I^a f^a(i)\, \ell^a(i)}{I^b f^b(\sigma(i))\, \ell^b(i)},$$

with boundary conditions $\sigma(\underline{i}^a) = \bar{i}^b$ *and* $\sigma(\bar{i}^a) = \underline{i}^b$*, and the condition that, for any fixed* $U^a(\underline{i}^a)$ *and* $U^b(\underline{i}^b)$*, the pairs* $(\ell^a(\underline{i}^a), U^b(\bar{i}^b))$ *and* $(\ell^a(\bar{i}^a), U^b(\underline{i}^b))$ *solve the unconstrained problems for the* $(\underline{i}^a, \bar{i}^b)$ *and* $(\bar{i}^a, \underline{i}^b)$ *markets, respectively:*

$$c = \max_{\lambda^a} m(\lambda^a, \ell^b(\underline{i}^a)) \left(u^a(\underline{i}^a, \bar{i}^b) + u^b(\bar{i}^b, \underline{i}^a) - \frac{U^a(\underline{i}^a)}{\lambda^a} - \frac{U^b(\bar{i}^b)}{\ell^b(\underline{i}^a)} \right), \tag{33}$$

$$c = \max_{\lambda^b} m(\ell^a(\bar{i}^a), \lambda^b) \left(u^a(\bar{i}^a, \underline{i}^b) + u^b(\underline{i}^b, \bar{i}^a) - \frac{U^a(\bar{i}^a)}{\ell^a(\bar{i}^a)} - \frac{U^b(\underline{i}^b)}{\lambda^b} \right), \tag{34}$$

where $\ell^b(\underline{i}^a)$ *and* $\ell^a(\bar{i}^a)$ *are pinned down by the local ICs for types* $\bar{i}^b$ *and* $\bar{i}^a$*, respectively.*

Proof of Characterization 5. The first four lines follow from the same steps as in the proof of Characterization 4; the derivation does not use the sign of σ'. For market clearing, consistency of the equilibrium measures gives $d\nu^b/d\nu^a = \ell^a/\ell^b$ on the active curve. Equating accumulated side-b agents used by side-a types in $[\underline{i}^a, i]$ with available supply in $[\sigma(i), \bar{i}^b]$:

$$I^b\big(F^b(\bar{i}^b) - F^b(\sigma(i))\big) = \int_{\underline{i}^a}^{i} \frac{\ell^a(x)}{\ell^b(x)}\, I^a\, dF^a(x).$$

Differentiating yields the market-clearing equation. ■